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Non-commutative standard polynomials applied to matrices

Denis Serre

UMPA, UMR CNRS–ENS Lyon # 5669, École Normale Supérieure de Lyon, 46, allée d'Italie, F-69364 Lyon cedex 07, France

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The Amitsur–Levitski Theorem tells us that the standard polynomial in $2n$ non-commuting indeterminates vanishes identically over the matrix algebra $\mathbf{M}_n(K)$. For $K = \mathbb{R}$ or $\mathbb{C}$ and $2 \le r \le 2n-1$, we investigate how big $\mathcal{S}_r(A_1, \dots, A_r)$ can be when $A_1, \dots, A_r$ belong to the unit ball. We privilege the Frobenius norm, for which the case $r = 2$ was solved recently by several authors. Our main result is a closed formula for the expectation of the square norm. We also describe the image of the unit ball when $r = 2$ or 3 and $n = 2$.

1. The problem. First results

Let $r \ge 2$ be an integer. The standard polynomial in r *non-commuting* indeterminates $x_1, \dots, x_r$ is defined as usual by

$$\mathcal{S}_r(x_1, \dots, x_r) := \sum \{\epsilon(\sigma) x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(r)} \; : \; \sigma \in \mathfrak{S}_r\},$$

E-mail address: denis.serre@ens-lyon.fr.

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where $\mathfrak{S}_r$ is the symmetric group in r symbols and ϵ is the signature. Each monomial is a word in the letters x_j, affected by a sign ± 1. Despite its superficial similarity with the determinant of $r \times r$ matrices, $\mathcal{S}_r$ is a completely different object: on the one hand, its arguments are non-commuting indeterminates, on the other hand, there are only r indeterminates instead of the r^2 entries of a matrix. We list here elementary properties of $\mathcal{S}_r$:

1. $\mathcal{S}_r$ is alternating.
2. $\mathcal{S}_{r+1}(x_1, \dots, x_{r+1}) = \sum_i (-1)^{i+1} x_i \mathcal{S}_r(\hat{x}_i)$, where $\hat{x}_i := (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{r+1})$.
3. If r is even, and an x_i commutes with all other x_j's, then $\mathcal{S}_r(x_1, \dots, x_r) = 0$. Mind that this is false if r is odd.

The first polynomial $x_1x_2 - x_2x_1$ of the list is the commutator. When applied to the elements of an algebra $\mathcal{A}$, it leads us to distinguish between commutative and non-commutative algebras. More generally, the polynomials $\mathcal{S}_r$ measure somehow the degree of non-commutativity of a given algebra. A classical theorem tells us that for a given matrix $A \in \mathbf{M}_n(\mathbb{C})$, the commutator $\mathcal{S}_2$ vanishes identically over the algebra $\langle A, A^* \rangle$ (in other words, A is normal) if and only if A is unitarily diagonalizable. It is less known that $\mathcal{S}_{2\ell}$ vanishes identically over the algebra $\langle A, A^* \rangle$ if and only if A is unitarily blockwise diagonalizable, where the diagonal blocks have at most size $\ell \times \ell$; see Exercise 324 in [10].

In addition, we have the theorem of Amitsur and Levitski [2], of which an elegant proof has been given by Rosset [8].

Theorem 1.1 *(Amitsur–Levitski). Let K be a field (a commutative one, needless to say). The standard polynomial $\mathcal{S}_{2n}$ of degree $2n$ vanishes identically over $\mathbf{M}_n(K)$. However the standard polynomials of degree less than $2n$ do not vanish identically.*

In the sequel, we focus on the algebra $\mathbf{M}_n(K)$ ($K = \mathbb{R}$ or $\mathbb{C}$) of real or complex matrices. A norm over $\mathbf{M}_n(K)$ is submultiplicative if it satisfies $\|AB\| \le \|A\| \, \|B\|$. The main examples are operator norms

$$\|A\| := \sup_{x \in K^n,\, x \neq 0} \frac{|Ax|}{|x|},$$

where $|\cdot|$ is a given norm over K^n. One often says that $\|\cdot\|$ is *induced* by $|\cdot|$. In particular, $\|\cdot\|_2$ is the norm induced by the standard Euclidean/Hermitian norm. We are also interested in the Frobenius norm

$$\|A\|_F := \sqrt{\sum_{i,j} |a_{ij}|^2},$$