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An arithmetic–geometric mean inequality for products of three matrices

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ABSTRACT

Consider the following noncommutative arithmetic–geometric mean inequality: Given positive-semidefinite matrices $\boldsymbol{A}_1, \dots, \boldsymbol{A}_n$, the following holds for each integer $m \leq n$:

$$\frac{1}{n^m} \sum_{j_1, j_2, \dots, j_m = 1}^{n} |||\boldsymbol{A}_{j_1} \boldsymbol{A}_{j_2} \dots \boldsymbol{A}_{j_m}||| \geq \frac{(n-m)!}{n!} \sum_{\substack{j_1, j_2, \dots, j_m = 1 \\ \text{all distinct}}}^{n} |||\boldsymbol{A}_{j_1} \boldsymbol{A}_{j_2} \dots \boldsymbol{A}_{j_m}|||,$$

where $|||\cdot|||$ denotes a unitarily invariant norm, including the operator norm and Schatten p-norms as special cases. While this inequality in full generality remains a conjecture, we prove that the inequality holds for products of up to three matrices, $m \leq 3$. The proofs for $m = 1, 2$ are straightforward; to derive the proof for $m = 3$, we appeal to a variant of the classic Araki–Lieb–Thirring inequality for permutations of matrix products.

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1. Introduction

The arithmetic–geometric mean (AMGM) inequality says that for any sequence of n non-negative real numbers $x_1, x_2, \ldots, x_n$, the arithmetic mean is greater than or equal to the geometric mean:

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \quad \geq \quad (x_1 x_2 \ldots x_n)^{1/n}.$$

This can be viewed as a special case ($m = n$) of Maclaurin's inequality:

Proposition 1.1. *If $x_1, \ldots, x_n$ are positive scalars and $m \geq n$ then it holds that*

$$\frac{1}{n^m} \sum_{j_1, j_2, \ldots, j_m = 1}^{n} x_{j_1} x_{j_2} \ldots x_{j_m} \quad \geq \quad \frac{1}{\binom{n}{m}} \sum_{\substack{\Lambda \subset [n]; \\ |\Lambda| = m}} x_{j_1} x_{j_2} \ldots x_{j_m}.$$

See [8] for more details. With a slight abuse of notation, we will refer to both of the above inequalities as AMGM inequalities.

Several noncommutative extensions of Proposition 1.1 have been proven for inequalities involving the product of two matrices. For an overview of these results, we refer the reader to [5]. These inequalities are often stated for a general unitarily invariant (UI) norm. Recall that a norm $|||\cdot|||$ on $M(d)$, the space of complex $d \times d$ matrices, is said to be unitarily invariant if for all $\boldsymbol{X}, \boldsymbol{U} \in M(d)$ with $\boldsymbol{U}$ unitary, one has $|||\boldsymbol{UX}||| = |||\boldsymbol{XU}||| = |||\boldsymbol{X}|||$. Examples of UI norms are the Schatten p-norms (including the operator norm and the Hilbert–Schmidt norm) and the Ky Fan k-norms. More generally, every UI norm is a symmetric gauge function of the singular values [2]. The first AMGM inequality for products of two matrices appeared in [4]: If $\boldsymbol{A}$ and $\boldsymbol{B}$ are compact operators on a separable Hilbert space, then

$$2|||\boldsymbol{A}^*\boldsymbol{B}||| \leq |||\boldsymbol{A}\boldsymbol{A}^* + \boldsymbol{B}\boldsymbol{B}^*|||.$$

The paper [3] extended this result, showing that for arbitrary $d \times d$ matrices $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{X}$, and for every unitarily invariant norm,

$$2|||\boldsymbol{A}^*\boldsymbol{X}\boldsymbol{B}||| \leq |||\boldsymbol{A}\boldsymbol{A}^*\boldsymbol{X} + \boldsymbol{X}\boldsymbol{B}\boldsymbol{B}^*|||.$$

Most closely related to the results here, Kosaki [10] showed that for positive-semidefinite matrices $\boldsymbol{A}$ and $\boldsymbol{B}$, and for $1/p + 1/q = 1$,

$$|||\boldsymbol{A}\boldsymbol{X}\boldsymbol{B}||| \leq \frac{1}{p}|||\boldsymbol{A}^p\boldsymbol{X}||| + \frac{1}{q}|||\boldsymbol{X}\boldsymbol{B}^q|||.$$

In the special case $\boldsymbol{X} = \mathrm{Id}$ and $p = q = 2$, applying this inequality and averaging with respect to the order of $\boldsymbol{A}$ and $\boldsymbol{B}$, reproduces our result for the case of two matrices.