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Square-zero factorization of matrices

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ABSTRACT

For given $m \times n$ matrices G and F over an arbitrary field $\mathbb{F}$, necessary and sufficient conditions (in terms of rank, amongst others) are presented for F to divide G with a square-zero quotient. These results are then used to extend the results of Novak [3] on square-zero factorization to matrices over an arbitrary field. The ranks that the square-zero factors can have are also investigated. Formulae are also presented by which these quotients can be constructed when this type of division is possible.

1. Introduction

A matrix $G \in M_{m\times n}(\mathbb{F})$, $\mathbb{F}$ an arbitrary field, is right divisible by $F \in M_{k\times n}(\mathbb{F})$ if $G = HF$ for some $H \in M_{m\times k}(\mathbb{F})$. F is called a right divisor (or right factor) of G, and H is called a right quotient of G and F (the remainder is always assumed to be zero). Similar definitions hold for left division. Since in this paper we shall be inter-

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ested in square-zero quotients, both G and F are assumed to be $m \times n$ unless stated otherwise.

We begin by presenting necessary and sufficient conditions (in terms of rank, amongst others) for F to be a divisor of G with a square-zero quotient. Since the similarity class of a square-zero matrix is completely determined by its rank, it is useful to also investigate which ranks the square-zero quotients can have. Formulae are presented by which square-zero quotients with every possible rank can be constructed. The preceding theory is then used to extend the results of Novak [3] on square-zero factorization to matrices over an arbitrary field. In particular, it is shown that

(i) an $m \times m$ matrix G over an arbitrary field $\mathbb{F}$ is a product of square-zero matrices if and only if $r(G) \leq \frac{m}{2}$, and three is the least number of factors required in general, and
(ii) G is a product of two square-zero matrices if and only if

$$n(G) - dim(N(G) \cap R(G)) \geq r(G)$$

equivalently,

$$G \text{ is similar to } O_{r(G)} \oplus G' \text{ for some matrix } G' \text{ over } \mathbb{F}.$$

We now fix some notation. For a given matrix $G \in M_{m\times n}(\mathbb{F})$, it will sometimes be useful to consider the linear mapping from $\mathbb{F}^n$ to $\mathbb{F}^m$ associated with G with respect to the standard bases for $\mathbb{F}^n$ and $\mathbb{F}^m$. This is also denoted by G (hence $G(v) = Gv$ for all $v \in \mathbb{F}^n$) since the intended meaning will be clear from the context. We denote the kernel (equivalently, null space) of G by $N(G)$, the image (equivalently, range or column space) of G by $R(G)$, and their respective dimensions by $n(G)$ and $r(G)$. The row space of G is denoted by $row(G)$. Similarity in $M_{n\times n}(\mathbb{F})$ is denoted by $\approx$. A left (resp., right) inverse of $G \in M_{m\times n}(\mathbb{F})$, if it exists, is denoted by G^L (resp., G^R). If $\mathbb{F} = \mathbb{C}$ or $\mathbb{F} = \mathbb{R}$, then the conjugate transpose of G is denoted by G^* (note that $G^* = G^T$ if $\mathbb{F} = \mathbb{R}$). The restriction of a linear mapping $T : V \to W$ to a subspace U of V is denoted by T_U. If $V = W$, then U is T-*invariant* if $T(U) \subseteq U$; hence T_U is a linear operator on U. For a real number r, let $\lfloor r \rfloor$ denote the largest integer less than or equal to r. The difference between two sets A and B is denoted by $A - B$, i.e. $A - B = \{a \in A : a \notin B\}$.

2. Matrix division with a square-zero quotient

Lemma 1. *Let $G, F \in M_{m\times n}(\mathbb{F})$, where $\mathbb{F}$ is a field, and let*

$$\mathbb{F}^m = R([G \quad F]) \oplus R(C) \text{ and } R(G) + R(F_{N(G)}) = R(G) \oplus R(B)$$

where both $B \in M_{m\times *}(\mathbb{F})$ and $C \in M_{m\times *}(\mathbb{F})$ are of full column rank. If H is a square-zero right quotient of G and F, then