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Convexity of a small ball under quadratic map

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ABSTRACT

We derive an upper bound on the size of a ball such that the image of the ball under quadratic map is strongly convex and smooth. Our result is the best possible improvement of the analogous result by Polyak [1] in the case of a quadratic map. We also generalize the notion of the joint numerical range of m-tuple of matrices by adding vector-dependent inhomogeneous term and provide a sufficient condition for its convexity.

1. Introduction and main result

1.1. Polyak convexity principle

Convexity is a highly appreciated feature which can drastically simplify analysis of various optimization and control problems. In most cases, however, the problem in ques-

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tion is not convex. In [1] Polyak proposed the following approach which proved to be useful in many applications [2,3]: to restrict the optimization or control problem to a small convex subset of the original set. More concretely, for a map $y_i = f_i(x)$ from $\mathbb{R}^n$ to $\mathbb{R}^m$, instead of the full image $\mathscr{F}(f) \equiv f(\mathbb{R}^n) = \{f(x) : x \in \mathbb{R}^n\}$, which is not necessarily convex, let us consider an image of a small ball $B_\varepsilon(x_0) = \{x : |x - x_0|^2 \leq \varepsilon^2\}$. For a regular point x_0 of $f_i(x)$ there is always small ε such that the image $f(B_\varepsilon(x_0))$ is convex. The underlying idea here is very simple: for any x from a small vicinity of a regular point x_0, where rank $(\partial f(x_0)/\partial x) = m$, the map $f(x)$ can be approximated by a linear map

$$y_i(x) - y_i(x_0) \simeq \left.\frac{\partial f_i}{\partial x^a}\right|_{x_0} (x - x_0)^a \ . \tag{1.1}$$

Since the linear map preserves strong convexity, so far the nonlinearities of $f(x)$ are small and can be neglected, the image of a small ball around x_0 will be convex. Reference [1] computes a conservative upper bound on $\varepsilon \leq \varepsilon_P$ in terms of the smallest singular value ν of the Jacobian $J(x_0) \equiv \left.\frac{\partial f}{\partial x}\right|_{x_0}$ and the Lipschitz constant L of the Jacobian $\partial f(x)/\partial x$ inside $B_\varepsilon(x_0)$,

$$\varepsilon_P^2 = \frac{\nu^2}{4L^2} \ . \tag{1.2}$$

The resulting image of $B_\varepsilon(x_0)$ satisfies the following two properties.

1. The image $f(B_\varepsilon(x_0))$ is strictly convex.
2. The pre-image of the boundary $\partial f(B_\varepsilon(x_0))$ belongs to the boundary $\partial B_\varepsilon(x_0) = \{x : |x - x_0|^2 = \varepsilon^2\}$. The interior points of $B_\varepsilon(x_0)$ are mapped into the interior points of $f(B_\varepsilon(x_0))$.

1.2. Local convexity of quadratic maps

In this paper we consider quadratic maps from $\mathbb{R}^n$ (or $\mathbb{C}^n$) to $\mathbb{R}^m$ of general form

$$f_i(x) = x^* A_i\, x - v_i^* x - x^* v_i \ , \tag{1.3}$$

defined through an m-tuple of symmetric (hermitian) $n \times n$ matrices A_i and an m-tuple of vectors $v_i \in \mathbb{R}^n$ (or $v_i \in \mathbb{C}^n$). Most of the results are equally applicable to both real $x \in \mathbb{R}^n$ and complex $x \in \mathbb{C}^n$ cases. The symbol * denotes transpose or hermitian conjugate correspondingly. Occasionally we will also use ^{T} to denote transpose for the explicitly real-valued quantities.

Applying the general theory of [1] toward (1.3) one obtains (1.2), where ν^2 is the smallest eigenvalues of the symmetric $m \times m$ matrix $\mathrm{Re}(v_i^* v_j)$ and the Lipschitz constant L for (1.3) can be defined through