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Cohomologically rigid solvable Lie algebras with a nilradical of arbitrary characteristic sequence

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ABSTRACT

It is shown that for a finite-dimensional solvable rigid Lie algebra $\mathfrak{r}$, its rank is upper bounded by the length of the characteristic sequence $c(\mathfrak{n})$ of its nilradical $\mathfrak{n}$. For any characteristic sequence $c = (n_1, \cdots, n_k, 1)$, it is proved that there exists at least a solvable Lie algebra $\mathfrak{r}_c$ the nilradical of which has this characteristic sequence and that satisfies the conditions $H^p(\mathfrak{r}_c, \mathfrak{r}_c) = 0$ for $p \leq 3$.

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1. Introduction

The rigidity problem of Lie algebras has been analyzed by many authors, either from the geometrical point of view, as well as from the cohomological perspective [1,2,12,10,11]. While for the case of finite-dimensional Lie algebras $\mathfrak{g}$ the vanishing of the second cohomology group $H^2(\mathfrak{g}, \mathfrak{g})$ ensures rigidity [16], the situation is radically different for the infinite-dimensional case, where non-equivalent notions of rigidity arise [9].
For the finite-dimensional case, the study of solvable rigid algebras has mainly been based on the properties of maximal tori of derivations of (specific) nilpotent algebras [2,7,12]. One question that has however not been addressed to is whether for any type of nilpotent Lie algebras there exists at least a solvable extension that is rigid. In this context, the type of a nilpotent Lie algebra $\mathfrak{n}$ is accurately described by an invariant $c(\mathfrak{n})$ called the characteristic sequence [1].

The objective of this work is to show that for any characteristic sequence $c = (n_1, \cdots, n_k, 1)$ with $n_i \geq 1$, there always exists a nilpotent Lie algebra $\mathfrak{n}_c$ that arises as the nilradical of a solvable rigid Lie algebra $\mathfrak{r}_c$. In particular, such a solvable extension has at most rank $k+1$, and is further a complete cohomologically rigid Lie algebra, i.e., satisfies $H^p(\mathfrak{r}_c, \mathfrak{r}_c) = 0$ for $p \leq 2$. This shows that rigidity of solvable Lie algebras is not restricted to the case of nilradical having maximal or minimal degrees of nilpotency, but rather covers all possible types of nilpotent Lie algebras.

Unless otherwise stated, any Lie algebra considered in this work is finite-dimensional and defined over $\mathbb{K} = \mathbb{R}, \mathbb{C}$.

1.1. Generalities

Given a Lie algebra $\mathfrak{g}$, we denote the Lie algebra formed by the derivations of $\mathfrak{g}$ by $Der(\mathfrak{g})$.

Definition 1. Let $\mathfrak{g}$ be a Lie algebra of dimension n. An external torus of derivations is any Abelian subalgebra of $Der(\mathfrak{g})$ the generators of which are semisimple.

As a consequence, maximal tori have the same dimension that we call the rank of $\mathfrak{g}$ and denote by $\mathrm{r}(\mathfrak{g})$. In the complex case, the Mal'cev theorem ensures that maximal tori are mutually conjugated [15].

If $\mathcal{L}^n$ denotes the class of n-dimensional Lie algebras $\mathfrak{g} = \left(\mathbb{K}^n, [\,,\,]_{\mathfrak{g}}\right)$ over $\mathbb{K}$, then the general linear group $GL(n, \mathbb{K})$ acts naturally on $\mathcal{L}^n$ by changes of basis:

$$(f.\mathfrak{g})(X, Y) = f^{-1}\left([f(X), f(Y)]_{\mathfrak{g}}\right), \; f \in GL(n, \mathbb{K}), \; X, Y \in \mathfrak{g}. \tag{1}$$

The orbit $\mathcal{O}(\mathfrak{g})$ of $\mathfrak{g}$ corresponds to those Lie algebras that are isomorphic to $\mathfrak{g}$.