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Spectral analysis of the anisotropic Steklov–Poincaré matrix

Mario Arioli [a,*], Daniel Loghin [b]

[a] Université de Toulouse-IRIT, 2 rue C. Camichel, Toulouse, 31000, France
[b] School of Mathematics, University of Birmingham, Birmingham, B15 2TT, United Kingdom

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ABSTRACT

In this work we analyse the Steklov–Poincaré (or interface Schur complement) matrix arising in a domain decomposition method in the presence of anisotropy. Our problem is formulated such that three types of anisotropy are being considered: refinements with high aspect ratios, uniform refinements of a domain with high aspect ratio and anisotropic diffusion problems discretized on uniform meshes. Our analysis indicates a condition number of the interface Schur complement with an order ranging from $\mathcal{O}(1)$ to $\mathcal{O}(h^{-2})$. By relating this behaviour to an underlying scale of fractional Sobolev spaces, we propose optimal preconditioners which are spectrally equivalent to fractional matrix powers of a discrete interface Laplacian. Numerical experiments to validate the analysis are included; extensions to general domains and non-uniform meshes are also considered.

* Corresponding author.
E-mail addresses: mario.arioli@gmail.com (M. Arioli), d.loghin@bham.ac.uk (D. Loghin).

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1. Background and motivation

Various analyses going back to the 1980s considered the discrete Steklov–Poincaré operator arising from two-domain decomposition methods for linear systems resulting from finite element discretizations of Poisson's problem on uniform meshes for simple polygonal shapes [5–7,3,4]. In all cases, the aim was to provide an eigenvalue analysis of the Schur complement (or capacitance) matrix arising in the case of a tractable decomposition. This effort resulted in the design of several interface preconditioners which are loosely speaking spectrally equivalent to a square-root Laplacian matrix associated with the interface. In turn, this fact is related to the boundedness of the Steklov–Poincaré operator when acting on fractional Sobolev spaces of index $\theta = 1/2$. Due to the complexity of the matrix square-root computation, the implementation of the preconditioners was expected to require fast Fourier transforms, under the assumption of mesh uniformity. Generalisations to the non-uniform case led to expensive implementations.

We consider in this work a generalisation of the contexts considered previously which allows for various types of anisotropy. In particular, we work with a scaled discrete Laplacian which can be associated with an anisotropic mesh or with an anisotropic diffusion operator. Both cases arise naturally in practice, not least in contexts that require fast, parallel algorithms for generating a solution. We provide an eigenvalue analysis in terms of both geometric and anisotropy parameters, which allows us to describe the dependence on anisotropy of the conditioning of the interface Schur complement. In particular, we show that the interface Schur complement is spectrally equivalent to more general fractional powers $\theta \in [0, 1]$ of a discrete Laplacian associated with the interface. We validate this conclusion with a range of numerical results. We also consider more general experiments inspired by the existing analysis.

2. The anisotropic Steklov–Poincaré matrix

Let $a \in \mathbb{R}_+$ and let $\Omega = (-1, 1) \times (-a, a)$. Consider the finite element solution of

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ \quad\; u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

using a partition of Ω into two equal domains separated by a horizontal boundary $\Gamma = \{(x, 0) : -1 \leq x \leq 1\}$ and subdivided uniformly into equal triangles with sides $h_x = 2/(n + 1)$, $h_y = 2a/(n + 1)$ with $n = 2m + 1$, $m \in N$. Let

$$T_k := \text{tridiag}[-1, 2, -1] \in \mathbb{R}^{k\times k}$$

denote a scaled FEM discretisation of $-\mathrm{d}^2/\mathrm{d}x^2$ on a mesh with k interior points and let $I_k \in \mathbb{R}^{k\times k}$ denote the identity matrix. With this notation, the discrete Laplacian matrix $L \in \mathbb{R}^{n^2\times n^2}$ is given by