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When local and global clustering of networks diverge

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A B S T R A C T

The average Watts–Strogatz clustering coefficient and the network transitivity are widely used descriptors for characterizing the transitivity of relations in real-world graphs (networks). These indices are bounded between zero and one, with low values indicating poor transitivity and large ones indicating a high proportion of closed triads in the graphs. Here, we prove that these two indices diverge for windmill graphs when the number of nodes tends to infinity. We also give evidence that this divergence occurs in many real-world networks, especially in citation and collaboration graphs. We obtain analytic expressions for the eigenvalues and eigenvectors of the adjacency and the Laplacian matrices of the windmill graphs. Using this information we show the main characteristics of two dynamical processes when taking place on windmill graphs: synchronization and epidemic spreading. Finally, we show that many of the structural and dynamical properties of a real-world citation network are well reproduced by the appropriate windmill graph, showing the potential of these graphs as models for certain classes of real-world networks.

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1. Introduction

The development of network theory has demanded the definition of several mathematical indices that characterize certain aspects of the topology of these usually giant graphs [1–3]. In a seminal paper published in 1998 Watts and Strogatz [4] defined an index nowadays known as the *Watts–Strogatz clustering coefficient* of a given node in a graph. This index accounts for the ratio of the number of triangles in which the corresponding node takes place to the number of potential triangles involving that node (see further for a formal definition). The clustering coefficient is bounded between zero and one. An index close to zero indicates that the relative number of transitive relations involving that node is low. A clustering coefficient close to one indicates that this node is involved in as many transitive relations as possible. When studying complex real-world networks it is very frequent to report the average Watts–Strogatz (WS) clustering coefficient as a characterization of how globally clustered a network is. Bollobás [5] has remarked that: "*this kind of 'average of an average' is often not very informative*".

An alternative way of characterizing the transitivity of a network is by taking the total number of triangles in the graph divided by the total number of triads existing in the graph. This index was put forward originally by Luce and Perry [6] and then rediscovered by Newman [7] in the context of complex networks. It is commonly referred to as the *transitivity index* of a (social) network (see for instance [8]). It is straightforward to realize that the transitivity index is also bounded between zero and one, with small values indicating poor transitivity and values close to one indicating a large one.

Due to the previous definitions of the WS average clustering coefficient and the transitivity index it is very surprising that there are simple graphs for which both indices diverge. In the Fig. 1 we give a couple of examples for which the WS average clustering coefficient tends asymptotically to zero while the transitivity index goes to one when the size of the graph tends to infinity. The first graph is known as the *friendship* (or Dutch windmill or n-fan) *graph* and the example was reported by the first time by Bollobás [5]. The second example was reported firstly by Estrada in [1] and these graphs were further named *agave graphs* by Estrada and Estrada-Vargas in a different context [9]. According to these examples, for a relatively large number of nodes, the WS average clustering coefficient tells us that the graphs are as transitive as possible, while the transitivity index is telling us exactly the contrary.

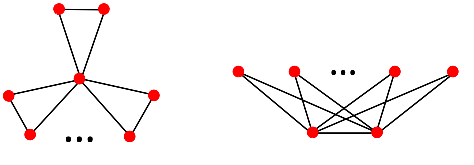

Fig. 1. Two examples of classes of graphs for which the Watts–Strogatz clustering coefficient and the graph transitivity diverge when the number of nodes tends to infinity.