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Equiangular lines and covers of the complete graph

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A R T I C L E I N F O

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A B S T R A C T

The relation between equiangular sets of lines in the real space and distance-regular double covers of the complete graph is well known and studied since the work of Seidel and others in the 70s. The main topic of this paper is to continue the study on how complex equiangular lines relate to distance-regular covers of the complete graph with larger index. Given a set of equiangular lines meeting the relative (or Welch) bound, we show that if the entries of the corresponding Gram matrix are prime roots of unity, then these lines can be used to construct an antipodal distance-regular graph of diameter three. We also study in detail how the absolute (or Gerzon) bound for a set of equiangular lines can be used to derive bounds of the parameters of abelian distance-regular covers of the complete graph.

1. Introduction

We explore the rich relation between equiangular lines in a real or complex vector space and covers of the complete graph. In our journey, we will link these concepts to other structures and translate properties across the different topics.

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Equiangular lines have been studied for a long a time, but there has been a recent surge in interest since the connection with quantum information theory was established (see, for example, Appleby [1] or Scott and Grassl [13]). A set of equiangular lines meeting the so-called *absolute (or Gerzon) bound* is also known as a *symmetric, informationally complete, positive operator valued measure (SIC-POVM)*, and the problem of constructing SIC-POVMs is a major problem both in quantum information theory and in combinatorics. Also another important connection is the well known correspondence between equiangular lines and Seidel matrices, and that a set of lines meets the so-called *relative (or Welch) bound* if and only if the corresponding Seidel matrix has only two distinct eigenvalues. These structures are also studied in frame theory, and sets of lines meeting the relative bound are called *equiangular tight frames*. In the electrical engineering literature, such sets are known as Welch bound equality sequences (see Wayne [15]).

In this paper, we study the relation between Seidel matrices and simple graphs that are covers of the complete graph. If a Seidel matrix has only two eigenvalues and if its entries are prime roots of unity, then we will show that such a matrix implies the existence of distance-regular covers of the complete graph whose automorphism group satisfies certain properties, namely, that the covers are cyclic (in the sense defined by Godsil and Hensel [6]). This is a natural generalization of the well known correspondence between real tight frames and regular two-graphs.

Furthermore, we will use this relation to derive bounds on the defining parameters of such graphs using the absolute bound for equiangular lines. It turns out that the existence of certain graphs could give d^2 equiangular lines in $\mathbb{C}^d$. We give the parameter sets of these graphs, and analyze the real case similarly.

2. Background on lines

A set of lines spanned by unit vectors $x_1, \ldots, x_n$ in $\mathbb{C}^d$ (or $\mathbb{R}^d$) is a set of *complex (or real) equiangular lines* if there is $\alpha \in \mathbb{R}$ such that, for all i and j,

$$|\langle x_i, x_j \rangle| = \alpha.$$

We will call α the *angle* between two lines.

Upon associating each line determined by x_i with the corresponding orthogonal projection given by $x_i x_i^*$, a set of equiangular lines is precisely the same thing as a 1-regular quantum design of degree 1. Such design will be called a *tight frame* if

$$x_1 {x_1}^* + \ldots + x_d {x_d}^* = \frac{n}{d} I.$$

Theorem 2.1 *(Relative bound). (See van Lint and Seidel [14].) If there is a set of n equiangular lines in dimension d, and if the angle of the set is α, then*

$$\alpha^2 \geq \frac{n-d}{(n-1)d}.$$

Equality holds if and only if the set of lines corresponds to a tight frame.