Linear Algebra and its Applications 488 (2016) 320–349

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Cayley-type graphs for group–subgroup pairs

Cid Reyes-Bustos

Graduate School of Mathematics, Kyushu University, 744 Motooka, Nishi-ku, Fukuoka, 819-0395, Japan

ARTICLE INFO

Article history:
Received 17 May 2015
Accepted 24 September 2015
Submitted by R. Brualdi

MSC:
05C25
05C50
20C99

Keywords:
Cayley graph
Bipartite graph
Ramanujan graph
Regular graph spectrum
Linear code

ABSTRACT

In this paper we introduce a Cayley-type graph for group–subgroup pairs (G, H) and certain subsets S of G. We present some elementary properties of such graphs, including connectedness, degree and partition structure, and vertex-transitivity, relating these properties with those of the underlying group–subgroup pair. From the properties of the underlying structures, some of the eigenvalues can be determined, including the largest eigenvalue of the graph. We present a sufficient condition on the group–subgroup pair (G, H) and the size of S that results on bipartite Ramanujan graphs. Among those Ramanujan graphs there are graphs that cannot be obtained as Cayley graphs. As another application, we propose the use of group–subgroup pair graphs to model linear error-correcting codes.

1. Introduction

For a group G and a symmetric subset S of G (i.e., $S^{-1} = S$), the Cayley graph $\mathcal{G}(G, S)$ is the graph whose vertices are the elements of G and where $g, h \in G$ are adjacent if $g^{-1}h$ is in S. A considerable amount of information about the graph can be determined from the properties of the group and the subset. Cayley graphs have been widely studied and

E-mail address: c-reyes@math.kyushu-u.ac.jp.

http://dx.doi.org/10.1016/j.laa.2015.09.049

their applications are well known, such as the construction of expander and Ramanujan graphs [8].

The aim of this paper is to introduce a new type of graph, constructed from a group G, a subgroup $H \subset G$, and a subset $S \subset G$, that is a generalization of Cayley graphs, study its basic properties and present some applications.

Definition 1.1. Let G be a group, H be a subgroup of G and S be a subset of G such that $S \cap H$ is a symmetric subset of H. The **Group–Subgroup Pair Graph** $\mathcal{G}(G, H, S)$ is the undirected graph with vertices G and simple edges given by

$$(h, hs) \qquad \forall h \in H, \quad \forall s \in S.$$

We use the term **pair-graph** as a synonymous for group–subgroup pair-graph.

When the group and the subgroup coincide the definition reduces to that of a Cayley graph. The motivation for the definition of the pair-graph comes from a recent paper [4] on the extension of the group determinant for group–subgroup pairs using the wreath determinant arising from the invariant theory of the α-determinant [5]. A note on the connection between the two concepts is provided in Appendix A.

In Section 2 we give basic examples and discuss the degree of the vertices of group–subgroup pair graphs. The degree structure of the resulting graphs depends on the relation of the generating subset with the cosets of the subgroup, as detailed in Proposition 2.6. For instance, group–subgroup pair graphs are not regular in general, but there are certain interesting properties on the degree structure of the graph.

Two important structural questions about a graph, especially for applications, are whether the graph is connected and if it is bipartite. In Section 3 we describe how the connectedness of the pair-graph is equivalent to two conditions. Namely, that

$$\langle H \cap ((S \cap H) \cup (S - H)(S - H)^{-1}) \rangle = H$$

and that the subset S contains representatives of the cosets of H. On the other hand, the condition $S \cap H = \emptyset$ is sufficient for the resulting graph to be bipartite. In general, the existence of a homomorphism $\chi : G \to \{-1, 1\}$ such that $\chi(S) = \{-1\}$ ensures that the Cayley graph $\mathcal{G}(G, S)$ is bipartite. In Theorem 4.5 we present an analogous condition for pair-graphs with respect to a homomorphism $\chi : H \to \{-1, 1\}$.

The structural properties of graphs according to the choice of subset S and the index of the subgroup are shown in Fig. 1.

Group–subgroup pair graphs contain a subset of eigenvalues that is apparent from the properties of the group G, subgroup H and the generating set S. When the pair-graph is k-regular, this set includes the trivial eigenvalue $\mu = k$. In the general case, we also present a lower bound for the multiplicity of the zero eigenvalue (see Proposition 5.4). The description of these apparent eigenvalues and their eigenfunctions is given in Section 5.