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Maximizing the spectral radius of k-connected graphs with given diameter ☆

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ABSTRACT

The spectral radius of a graph is the largest eigenvalue of its adjacency matrix. P. Hansen and D. Stevanović (2008) [9] determined the graphs with maximum spectral radius among all connected graphs of order n with diameter D. In this paper, we generalize this result to k-connected graphs of order n with diameter D.

1. Introduction

All graphs in this article are finite, simple and undirected. Let $G = (V(G), E(G))$ be a graph of order n with size m, where the vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. We denote $N_G(v)$ (or $N(v)$ for short) as the set of neighbours of v in G, and $|N_G(v)|$ as the degree of v. For $i \in \{1, 2, \ldots, n\}$, let $d_i = d_G(v_i) = |N_G(v_i)|$. Moreover, the maximum and

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minimum degrees of G are denoted by Δ and δ, respectively. Let $X \subseteq V(G)$, the graph $G - X$ is obtained from G by deleting all the vertices in X and their incident edges. A graph G is called *k-connected* (for $k \in \mathbb{N}$) if $|V(G)| > k$ and $G - X$ is connected for every set $X \subseteq V(G)$ with $|X| < k$. The greatest integer k such that G is k-connected is the *connectivity* $\kappa(G)$ of G. For two distinct vertices u and v, the *distance* between u and v is the length of a shortest path that contains u and v. And the *diameter* of G, denoted by $D(G)$ (or D for short), is the maximum distance among every pair of distinct vertices of G. All notations undefined in this article are referred to the book [2].

For a graph G, let $A(G)$ (or simply A) be the adjacency matrix of G according to the list of vertices $\{v_1, v_2, \ldots, v_n\}$. The *spectrum* of G is the set of eigenvalues of $A(G)$ together with their multiplicities. Since $A(G)$ is a real symmetric matrix, all its eigenvalues are real. Moreover, the largest eigenvalue of $A(G)$, denoted by $\lambda_1(G)$ (or simply λ_1), is called the *spectral radius* of G. For a connected graph G, $A(G)$ is a nonnegative irreducible matrix. Hence, by the Perron–Frobenius Theorem, the multiplicity of $\lambda_1(G)$ is one and there exists a positive eigenvector $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^T$, which is called the *Perron vector* of $A(G)$. The Perron vector satisfies $\sum_{i=1}^{n} x_i^2 = 1$ and the *eigenvalue equation* $A(G)\boldsymbol{x} = \lambda_1(G)\boldsymbol{x}$. Therefore

$$\lambda_1(G) = \boldsymbol{x}^T A(G)\boldsymbol{x} = 2 \sum_{v_i v_j \in E(G)} x_i x_j. \tag{1.1}$$

In addition, $\lambda_1(G)$ can be characterized by the Rayleigh quotient

$$\lambda_1(G) = \sup_{||\boldsymbol{x}||_2 = 1} \boldsymbol{x}^T A(G)\boldsymbol{x}. \tag{1.2}$$

Please refer to [3,7,13] for previous results about spectrum of graphs.

Brualdi and Solheid [4] proposed the following problem concerning the spectral radius of graphs: Given a set $\mathcal{G}$ of graphs, find an upper bound for the spectral radius of graphs in $\mathcal{G}$ and characterize the graphs for which the maximal spectral radius is attained. For many different sets $\mathcal{G}$, this problem has been solved (see [1,4–6] for details). Let $\mathcal{G}_n^D$ be the set of connected graphs of order n with diameter D. P. Hansen and D. Stevanović [9] and E.R. van Dam [8] solved the problem on $\mathcal{G}_n^D$ using two different methods. Firstly, we define a bug as follows.

A bug Bug_{p,q_1,q_2} is a graph obtained from a complete graph K_p by deleting an edge uv and attaching paths P_{q_1} and P_{q_2} at u and v, respectively. A bug is balanced if $|q_1 - q_2| \leq 1$. The following is their main result.

Theorem 1.1. *(See [9].) Among all connected graphs of order n with diameter D, the maximal spectral radius is attained by*

$$\begin{cases} \text{a complete graph } K_n, & \text{when } D = 1, \\ \text{a balanced bug } Bug_{n-D+2,\lceil D/2\rceil,\lfloor D/2\rfloor}, & \text{when } D \geq 2. \end{cases}$$