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Quotient-polynomial graphs

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A B S T R A C T

As a generalization of orbit-polynomial and distance-regular graphs, we introduce the concept of a quotient-polynomial graph. In these graphs every vertex u induces the same regular partition around u, where all vertices of each cell are equidistant from u. Some properties and characterizations of such graphs are studied. For instance, all quotient-polynomial graphs are walk-regular and distance-polynomial. Also, we show that every quotient-polynomial graph generates a (symmetric) association scheme.

1. Introduction and preliminaries

As expected, the most interesting combinatorial structures are those bearing some kind of symmetry and/or regularity. In fact, in general, high symmetry implies high regularity, but the converse does not necessarily hold. Moreover, symmetric structures suggest definitions of new structures obtained, by either relaxing the conditions of symmetry, or

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having the same regularity properties as the original ones. In turn, the latter can give rise to new definitions by relaxing the mentioned symmetry conditions. In graph theory, a good example of the above are the distance-transitive graphs, with automorphism group having orbits constituted by all vertices at a given distance. Attending to their symmetry, related concepts are the vertex-symmetric, symmetric, and orbit polynomial graphs [2,3]. Besides, concerning regularity, distance-transitive graphs can be generalized to distance-regular graphs [1,4], distance-polynomial graphs, and degree-regular graphs [14]. In this paper, we introduce the concept of a quotient-polynomial graph, which could be thought of as the regular counterpart of orbit polynomial graphs. In a quotient-polynomial graph, every vertex u induces the same regular partition around u, with the additional condition that all vertices of each cell are equidistant from u. Some properties and characterizations of such graphs are studied. For instance, all quotient-polynomial graphs are walk-regular and distance-polynomial, and we provide a characterization of those distance-polynomial graphs which are distance-regular. Also, we show that every quotient-polynomial graph generates a (symmetric) association scheme.

Throughout this paper, Γ denotes a (connected) graph with vertex set V, edge set E, and diameter D. For every $u \in V$ and $i = 0, \dots, D$, let $\Gamma_i(u)$ denote the set of vertices at distance i from u, with $\Gamma(u) = \Gamma_1(u)$, and let $\boldsymbol{e}_u$ be the characteristic (u-th unitary) vector of $\Gamma_0(u)$. The *eccentricity* of u, denoted by $\varepsilon(u)$, is the maximum distance between u and any other vertex v of Γ. Let $\boldsymbol{A}_i$ be the i-th distance matrix, so that $\boldsymbol{A} = \boldsymbol{A}_1$ is the adjacency matrix of Γ, with spectrum $\operatorname{sp}\Gamma = \{\lambda_0^{m_0}, \dots, \lambda_d^{m_d}\}$, where $\lambda_0 > \lambda_1 > \cdots > \lambda_d$, and the superscripts $m_i = m(\lambda_i)$ stand for the multiplicities. Let $\boldsymbol{E}_j$, $j = 0, \dots, d$, be the minimal idempotents representing the orthogonal projections onto the λ_j-eigenspaces. Let $\mathcal{A}(\Gamma) = \mathbb{R}_d[\boldsymbol{A}]$ be the *adjacency algebra* of Γ, that is, the algebra of all polynomials in $\boldsymbol{A}$ with real coefficients.

Following Fiol, Garriga and Yebra [8,9], the uv-entry of $\boldsymbol{E}_j$ is referred to as the *crossed* (uv-)*local multiplicity* of the eigenvalue λ_j, and it is denoted by $m_{uv}(\lambda_j)$. In particular, for a regular graph on n vertices, $\boldsymbol{E}_0 = \frac{1}{n}\boldsymbol{J}$ and, hence, $m_{uv}(\lambda_0) = 1/n$ for every $u, v \in V$. Since $\boldsymbol{A}^\ell = \sum_{j=0}^d \lambda_j^\ell \boldsymbol{E}_j$, the number of walks of length ℓ between two vertices u, v is

$$a_{uv}^{(\ell)} = (\boldsymbol{A}^\ell)_{uv} = \sum_{j=0}^{d} m_{uv}(\lambda_j)\lambda_j^\ell \qquad (\ell \geq 0). \tag{1}$$

In particular, the (u-)*local multiplicities* are $m_u(\lambda_i) = \|\boldsymbol{E}_i\boldsymbol{e}_u\|^2 = (\boldsymbol{E}_i)_{uu}$, $i = 0, \dots, d$, and satisfy $\sum_{i=0}^d m_u(\lambda_i) = 1$ and $\sum_{u\in V} m_u(\lambda_i) = m_i$, $i = 0, \dots, d$.

A graph Γ with diameter D is called *h-punctually walk-regular*, for some $h = 0, \dots, D$, when the number of walks $a_{uv}^{(\ell)}$ for any pair of vertices u, v at distance h only depends on ℓ. From the above, this means that the crossed local multiplicities $m_{uv}(\lambda_j)$ only depend on λ_j and we write them as $m_h(\lambda_j)$ (see Dalfó, Van Dam, Fiol, Garriga and Gorissen [5] for more details). Notice that, in particular, a 0-punctually walk-regular graph is the same as a walk-regular graph, a concept introduced by Godsil and McKay [11].