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An infinite family of excluded minors for strong base-orderability

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We discuss a conjecture of Ingleton on excluded minors for base-orderability, and, extending a result he stated, we prove that infinitely many of the matroids that he identified are excluded minors for base-orderability, as well as for the class of gammoids. We prove that a paving matroid is base-orderable if and only if it has no $M(K_4)$-minor. For each $k \geq 2$, we define the property of k-base-orderability, which lies strictly between base-orderability and strong base-orderability, and we show that k-base-orderable matroids form what Ingleton called a complete class. By generalizing an example of Ingleton, we construct a set of matroids, each of which is an excluded minor for k-base-orderability, but is $(k-1)$-base-orderable; the union of these sets, over all k, is an infinite set of base-orderable excluded minors for strong base-orderability.

1. Introduction

Basis-exchange properties are of long-standing interest in matroid theory (see Kung's survey [13]). Condition (BE) in the following definition of a matroid is a simple basis-exchange property: a matroid M is an ordered pair $(\mathcal{B}, E(M))$ where $E(M)$ is

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a finite set and $\mathcal{B}$ is a non-empty collection of subsets of $E(M)$ (the bases) such that

(BE) *if* $B_1, B_2 \in \mathcal{B}$ *and* $x \in B_1 - B_2$*, then there is some* $y \in B_2 - B_1$ *so that* $(B_1 - x) \cup y \in \mathcal{B}$.

Several basis-exchange properties that appear to be stronger (such as symmetric basis-exchange) hold in all matroids and so are equivalent to (BE); see [4,7,9,23]. In contrast, our work here is motivated by the following basis-exchange properties that are not possessed by all matroids.

Definition 1.1. A matroid M is *base-orderable* if, given any two bases B_1 and B_2, there is a bijection $\sigma: B_1 \to B_2$ such that for every $x \in B_1$, both $(B_1-x)\cup\sigma(x)$ and $(B_2-\sigma(x))\cup x$ are bases.

A matroid M is *strongly base-orderable* if, given any two bases B_1 and B_2, there is a bijection $\sigma: B_1 \to B_2$ such that for every $X \subseteq B_1$,

(*) $(B_1 - X) \cup \sigma(X)$ is a basis, and
(**) $(B_2 - \sigma(X)) \cup X$ is a basis.

To the best of our knowledge, the notion of base-orderability first appeared in [4,6]. Brualdi and Scrimger [6] showed that all transversal matroids are strongly base-orderable (and hence base-orderable). The property of base-orderability appeared (without the term) in Brualdi [4] as a natural strengthening of the basis-exchange properties discussed there.

Not all matroids are base-orderable; in particular, the cycle matroid $M(K_4)$ is not. We denote the class of base-orderable matroids by $\mathcal{BO}$ and that of strongly base-orderable matroids by $\mathcal{SBO}$. (In this paper, by a *class* of matroids we mean a set of matroids that is closed under isomorphism.) Clearly, $\mathcal{SBO} \subseteq \mathcal{BO}$. Ingleton [10] gave an example that shows that this containment is proper. In Section 9 we generalize his example; we construct an infinite collection of excluded minors for strong base-orderability, each of which is base-orderable.

It is easy to show that the class of base-orderable matroids is minor-closed, but describing its excluded minors remains an open problem. In Section 5, we discuss a conjecture of Ingleton on the excluded minors. Much of our work arose by exploring ideas in Ingleton's paper [10], to which we owe a great debt. A number of our results and constructions grew from seeds in that paper, which, while providing a wealth of intriguing ideas, contains few proofs. To give a more complete account of this topic, we also offer proofs of some of the assertions that Ingleton made, either without proof or with a minimal sketch of the proof. In Section 3, we lay the groundwork for Section 5 and also prove that a paving matroid is base-orderable if and only if it has no $M(K_4)$-minor. In Section 4, we review cyclic flats, which we use extensively thereafter. In Section 8, we prove a special case of