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Polynomial zigzag matrices, dual minimal bases, and the realization of completely singular polynomials

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ABSTRACT

Minimal bases of rational vector spaces are a well-known and important tool in systems theory. If minimal bases for two subspaces of rational n-space are displayed as the rows of polynomial matrices $Z_1(\lambda)_{k\times n}$ and $Z_2(\lambda)_{m\times n}$, respectively, then Z_1 and Z_2 are said to be *dual* minimal bases if the subspaces have complementary dimension, i.e., $k+m=n$, and $Z_1(\lambda)Z_2^T(\lambda)=0$. In other words, each $Z_j(\lambda)$ provides a minimal basis for the nullspace of the other. It has long been known that for any dual minimal bases $Z_1(\lambda)$ and $Z_2(\lambda)$, the row degree sums of Z_1 and Z_2 are the same. In this paper we show that this is the only constraint on the row degrees, thus characterizing the possible row degrees of dual minimal bases. The proof is constructive, making extensive use of a new class of sparse, structured

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Minimal indices
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polynomial matrices that we have baptized *zigzag* matrices. Another application of these polynomial zigzag matrices is the constructive solution of the following inverse problem for minimal indices: Given a list of left and right minimal indices and a desired degree d, does there exist a completely singular matrix polynomial (i.e., a matrix polynomial with no elementary divisors whatsoever) of degree d having exactly the prescribed minimal indices? We show that such a matrix polynomial exists if and only if d divides the sum of the minimal indices. The constructed realization is simple, and explicitly displays the desired minimal indices in a fashion analogous to the classical Kronecker canonical form of singular pencils.

1. Introduction

The notion of a minimal basis, formed by vectors with polynomial entries, of a rational vector subspace was made popular by the books of Wolovich [20] and Kailath [12], and by the paper of Forney [8], although all three of them cite earlier work for the basic ideas of these so-called *minimal polynomial bases*. The main contribution of these authors is twofold: they provided computational schemes for constructing a minimal basis from an arbitrary polynomial basis, and they showed the importance of this notion for multivariable linear systems. These systems could be modeled by rational matrices, polynomial matrices, or linearized state–space models, and had tremendous potential for solving analysis and design problems in control theory as well as in coding theory.

One such classical design problem was to show the relations between left and right coprime factorizations of a rational matrix $R(\lambda)$ of size $m \times k$:

$$D_\ell(\lambda)^{-1} N_\ell(\lambda) = R(\lambda) = N_r(\lambda) D_r(\lambda)^{-1},$$

where $D_\ell(\lambda)$, $N_\ell(\lambda)$, $N_r(\lambda)$, $D_r(\lambda)$ are all polynomial matrices, and $D_\ell(\lambda)$, $D_r(\lambda)$ are square and invertible. The coprimeness condition amounts to saying that the $m \times (m+k)$ and $k \times (m+k)$ matrices

$$Z_\ell(\lambda) := [D_\ell(\lambda), -N_\ell(\lambda)], \quad \text{and} \quad Z_r(\lambda) := [N_r(\lambda)^T, D_r(\lambda)^T]$$

have full row rank for all $\lambda \in \mathbb{C}$. It is easy to see that

$$D_\ell(\lambda)^{-1} N_\ell(\lambda) = N_r(\lambda) D_r(\lambda)^{-1} \qquad \text{if and only if} \qquad Z_\ell(\lambda) Z_r(\lambda)^T = 0,$$

which implies that the row spaces of $Z_\ell(\lambda)$ and $Z_r(\lambda)$ over the field of rational functions are "dual" to each other in the sense of Forney [8, Section 6]. In order to better understand the structure of these rational row spaces, one could then look for polynomial bases that