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Some geometric interpretations of quantum fidelity

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We consider quantum fidelity between two states ρ and σ, where we fix ρ and allow σ to be sent through a quantum channel. We determine the minimal fidelity where one minimizes over (a) all unital channels, (b) all mixed unitary channels, and (c) arbitrary channels. We derive results involving the minimal eigenvalue of ρ, which we can interpret as a convex combination coefficient. As a consequence, we give a new geometric interpretation of the minimal fidelity with respect to the closed, convex set of density matrices and with respect to the closed, convex set of quantum channels. We further investigate the geometric nature of fidelity by considering density matrices arising as normalized projections onto subspaces; in this way, fidelity can be viewed as a geometric measure of distance between two spaces. We give a connection between fidelity and the canonical (principal) angles between the subspaces.

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1. Introduction

The quantum fidelity $F(\rho, \sigma)$ is a measure of the distance between two quantum states ρ and σ that quantifies the accuracy of state transfer through a channel; the ideal case being a fidelity value of 1, which represents perfect state transfer. Physically, one begins with initial state ρ at time 0, and allows the quantum system to evolve over time. At time t, one measures the overlap of the two states ρ and σ; this overlap decreases over time due to the evolution and perturbation of the system.

Fidelity has been considered in the context of quantum communication via unmeasured and unmodulated spin chains which are used to transmit quantum states [2], and plays a role in quantum decision tree algorithms [10]. Geometric interpretations of fidelity have been given in [12,11] and elsewhere, however our approach and results are distinctly different from the literature at present.

Formally, we have the following definition:

Definition 1.1. Let ρ and σ be two $n \times n$ positive semidefinite matrices. The *(quantum) fidelity* between ρ and σ is

$$\begin{aligned} F(\rho, \sigma) &= \mathrm{Tr}(\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}) \\ &= \mathrm{Tr}(\sqrt{\sqrt{\sigma}\rho\sqrt{\sigma}}). \end{aligned}$$

Any positive semidefinite matrix has a unique positive square root, and so the quantum fidelity between two quantum states (density matrices) is well-defined and yields a non-negative real number.

The transition probability between two states, which is the square of the fidelity, was defined in [19], although the idea stems from two earlier papers [7,3] in a more general context. Jozsa [6] proposed four axioms that the transition probability (which he called fidelity) must satisfy (we have re-written these axioms in terms of fidelity)[1]

1. $0 \le F(\rho, \sigma) \le 1$ with $F(\rho, \sigma) = 1$ iff $\rho = \sigma$;
2. The fidelity is symmetric: $F(\rho, \sigma) = F(\sigma, \rho)$;
3. If $\rho = |\psi\rangle\langle\psi|$ is a pure state, then $F(\rho, \sigma) = \sqrt{\langle\psi|\sigma|\psi\rangle}$;
4. The fidelity is invariant under unitary transformations on the state space:

   $$F(U\rho U^\dagger, U\sigma U^\dagger) = F(\rho, \sigma) \quad \text{for any unitary } U,$$

   where † represents complex conjugate transposition.

[1] The language in the literature is not consistent. Some authors take the point of view of Jozsa: transition probability = fidelity = $F^2(\rho, \sigma)$, using the notation of Definition 1.1. They then call $F(\rho, \sigma)$ the square root fidelity.