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On characterization of operator monotone functions

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ABSTRACT

In this paper, we will show a new characterization of operator monotone functions by a matrix reverse Cauchy inequality.

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1. Introduction

Let M_n be the space of $n \times n$ complex matrices, M_n^h the self-adjoint part of M_n. For $A, B \in M_n^h$, the notation $A \leq B$ means that $B - A \in M_n^+$. The spectrum of a matrix $A \in M_n$ is denoted by $\sigma(A)$. For a real-valued function f of a real variable and a matrix $A \in M_n^h$, the value $f(A)$ is understood by means of the functional calculus for Hermitian matrices.

Taking an axiomatic approach, Kubo and Ando introduced the notions of connection and mean. A binary operation σ defined on the set of positive definite matrices is called a *connection* if

(i) $A \leq C, B \leq D$ imply $A\sigma B \leq B\sigma D$;
(ii) $C^*(A\sigma B)C \leq (C^*AC)\sigma(C^*BC)$;
(iii) $A_n \downarrow A$ and $B_n \downarrow B$ imply $A_n\sigma B_n \downarrow A\sigma B$.

If $I\sigma I = I$, then σ is called a *mean*.

For $A, B > 0$, the *geometric mean* $A\sharp B$ is defined by

$$A\sharp B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}.$$

The *harmonic* $A!B$ and *arithmetic* $A\nabla B$ means are defined $A!B = 2(A^{-1} + B^{-1})^{-1}$ and $A\nabla B = \dfrac{A+B}{2}$, respectively. A mean σ is called to be symmetric if $A\sigma B = B\sigma A$ for any pair of positive definite matrices A, B.

It is well-known that the arithmetic mean ∇ is the biggest among symmetric means. From the general theory of symmetric matrix means we know that $\nabla \geq \sigma$ and $\tau \geq !$.

For positive real numbers a, b, the arithmetic–geometric mean inequality (AGM) says that

$$\sqrt{ab} \leq \frac{a+b}{2}.$$

Hences, for a monotone increasing function f on $[0, \infty)$, we have

$$f(\sqrt{ab}) \leq f(\frac{a+b}{2}). \tag{1}$$

It is natural to ask that if inequality (1) holds for any pair of positive number a, b will the function f be monotone increasing on $[0, \infty)$? The answer is positive, and follows from the elementary fact that for any positive numbers $a \leq b$ there exist positive number x, y such that a is arithmetic mean and b is geometric mean of x, y.

The matrix version of above fact was investigated by Prof. T. Ando and Prof. F. Hiai [3]. They showed that the Cauchy inequality characterizes operator monotone functions, that means, if the following inequality holds