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Remarks on two recent results of Audenaert

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Audenaert recently obtained some new matrix norm inequalities in Audenaert (2015) [2,3]. In this note, we provide alternative proofs for these results.

1. Introduction

Let $\mathbb{M}_n$ be the set of $n \times n$ complex matrices. In [2], Audenaert obtained the following result

Theorem 1.1. *(See [2, Theorem 3.1].) For $i = 1, \dots, k$, let $A_i, B_i \in \mathbb{M}_n$ be positive semidefinite such that, for each i, A_i commutes with B_i. Then for any unitarily invariant norm,*

$$\left\| \sum_{i=1}^{k} A_i B_i \right\| \leq \left\| \left(\sum_{i=1}^{k} A_i^{1/2} B_i^{1/2} \right)^2 \right\| \leq \left\| \left(\sum_{i=1}^{k} A_i \right) \left(\sum_{i=1}^{k} B_i \right) \right\|.$$

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Recall that a norm $\|\cdot\|$ on $\mathbb{M}_n$ is unitarily invariant if $\|UAV\| = \|A\|$ for any unitary matrices $U, V \in \mathbb{M}_n$ and any $A \in \mathbb{M}_n$. In the sequel, $\|\cdot\|$ stands for any unitarily invariant norm. A special case (i.e., $k = 2$) of Theorem 1.1 confirms a conjecture of Hayajneh and Kittaneh (Conjecture 1.2 in [10]), in particular, the following conjectured inequality of Bourin [9]

Corollary 1.2. *Let $A_i, B_i \in \mathbb{M}_n$ be positive semidefinite and $p, q > 0$. Then*

$$\|A^{p+q} + B^{p+q}\| \le \|(A^p + B^p)(A^q + B^q)\|.$$

For $X \in \mathbb{M}_n$, the conjugate transpose of X is denoted by X^*. In another paper [3], Audenaert proved

Theorem 1.3. *For all $X, Y \in \mathbb{M}_n$, and all $q \in [0, 1]$,*

$$\|XY^*\|^2 \le \|qX^*X + (1-q)Y^*Y\| \; \|(1-q)X^*X + qY^*Y\|. \tag{1.1}$$

As explained in [3], inequality (1.1) interpolates between the Arithmetic–Geometric mean ($q = 1/2$) and Cauchy–Schwarz ($q = 0$) matrix norm inequalities.

The way that Audenaert proved Theorem 1.1 and Theorem 1.3 is original. Considering that alternative proofs may provide new perspectives to these elegant results, we take the chance to do it here.

2. Proof of Theorem 1.1

The following lemma is a special case of [6, Theorem 5] by taking $f(t) = t^r$, $t \in [0, \infty)$.

Lemma 2.1. *For $i = 1, \ldots, k$, let $A_i \in \mathbb{M}_n$ be positive semidefinite. Then*

$$\left\|\sum_{i=1}^k A_i^r\right\| \le \left\|\left(\sum_{i=1}^k A_i\right)^r\right\|, \qquad r > 1.$$

The inequality reverses for $0 \le r \le 1$.

We require a new result for our purpose. The block matrix $\begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$, where $A, B, X \in \mathbb{M}_n$, is positive partial transpose (i.e., PPT) if both $\begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$ and $\begin{pmatrix} A & X^* \\ X & B \end{pmatrix}$ are positive semidefinite.

Lemma 2.2. *Let $\begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$, where $A, B, X \in \mathbb{M}_n$, be PPT. Then*

$$\|X^*X\| \le \|AB\|. \tag{2.1}$$