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Deflation and projection methods applied to symmetric positive semi-definite systems

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ABSTRACT

Linear systems with a singular symmetric positive semi-definite matrix appear frequently in practice. This usually does not lead to difficulties for CG methods as long as these systems are consistent. However, the construction of a preconditioner, especially the construction of two-level and multilevel methods, becomes more complicated, since singular coarse grid matrices or Galerkin matrices may occur. Here we continue the work started in [21,22] where deflation is used for some special singular coefficient matrices. Here we show that deflation and other projection-type preconditioners can be applied to arbitrary singular problems without any difficulties. In each of these methods, a two-level preconditioner is involved where coarse-grid systems based on a singular Galerkin matrix should be solved. We prove that each projection operator consisting of a singular Galerkin matrix can be written as an operator with a nonsingular Galerkin matrix. Therefore many results that hold for nonsingular

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Galerkin matrices are also valid for problems with singular Galerkin matrices.

1. Introduction

The conjugate gradient (CG) method [6] is the method of choice for solving linear systems of the form

$$Ax = b, \quad A \in \mathbb{R}^{n\times n},$$

whose coefficient matrix, A, is sparse and symmetric positive semi-definite (SPSD). However, in order to speed up the convergence of CG, an efficient symmetric positive definite (SPD) preconditioner, M^{-1}, is needed. With a preconditioner, the system

$$M^{-1}Ax = M^{-1}b,$$

is then to be solved by CG. The preconditioner, M^{-1}, must be chosen such that $M^{-1}A$ has a more clustered spectrum or a smaller condition number than A. Furthermore, the system $My_2 = y_1$ must be cheap to solve relative to the improvement it provides in convergence rate. Nowadays, the design and analysis of preconditioners for CG is the main focus whenever a linear system with an SPSD coefficient matrix needs to be solved.

Stimulated by the numerical solution of PDEs and the problem of solving the resulting linear system, two-level or multi-level preconditioners have been developed successfully during the last decades. These preconditioners make use of the solution of smaller systems, which are also known as Galerkin systems or coarse-grid systems. In combination with traditional preconditioners, such as Jacobi and Gauss–Seidel, these preconditioners have become a powerful tool. Examples of these preconditioners are the geometric and algebraic multigrid methods (see, e.g., [25,27]) and domain decomposition methods with coarse-grid corrections, such as the BPS method [2,3] and the balancing Neumann–Neumann method [9–11]. In the same period, the so-called augmented Krylov-subspace methods were established [13]. These methods are also known as deflation-type methods [17,19].

At a first glance, all of the above described projection methods differ a lot in practice. However, from an abstract point of view, there are many common ingredients. Each of these methods use a projection of the form

$$P := I - AQ, \quad Q := ZE^{-1}Z^T, \quad E := Z^TAZ, \tag{1.1}$$

where we assume that $Z \in \mathbb{R}^{n\times r}$ is given such that E is nonsingular. In classical multilevel methods, Z is called a prolongation operator, while Z^T acts as a restriction operator. Moreover, E is known as the coarse-grid or Galerkin matrix. In the deflation methods,