

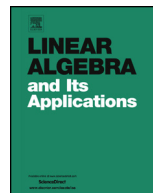


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Pairs of orthogonal projections with a fixed difference



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ABSTRACT

Let A be a self-adjoint contraction on a Hilbert space \mathcal{H} . In 2014, E. Andruchow proved that if A is a difference of two orthogonal projections, with non-trivial generic part, then there exist infinitely many pairs (P, Q) of orthogonal projections such that $A = P - Q$. In the present paper, we give a sufficient and necessary condition for A to be a difference of a pair of orthogonal projections. Then we give a characterization of all pairs (P, Q) of orthogonal projections such that $A = P - Q$. Moreover, we characterize the von Neumann algebra generated by such pairs (P, Q) , and consider the connected components of the set $\{(P, Q) : A = P - Q\}$.

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1. Introduction

The subject of pairs of orthogonal projections has been an active area in past decades. During this period, Kato [11], Dixmier [9], Halmos [10] and Davis [8] established many remarkable results. Recently, several related problems are studied by a number of au-

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thors [1,3–7,12]. In [8], C. Davis characterized the self-adjoint operators which are a difference of two orthogonal projections. Afterwards, E. Andruchow [2] proved that if A is a difference of two orthogonal projections, with non-trivial generic part, then there exist infinitely many pairs (P, Q) of orthogonal projections such that $A = P - Q$. It is therefore natural to ask for a description of all pairs (P, Q) of orthogonal projections satisfying $A = P - Q$. On the other hand, from the perspective of algebra, we need to know what is the von Neumann algebra generated by such pairs (P, Q) of orthogonal projections. We put $A_P = \{(P, Q) : A = P - Q\}$. The objective of this paper is twofold. The first one is to characterize A_P , the other is to get the von Neumann algebra generated by all pairs $(P, Q) \in A_P$ of orthogonal projections and its commutant.

Throughout this paper, let \mathcal{H} be a Hilbert space and $\mathcal{B}(\mathcal{H})$ the algebra of all bounded linear operators on \mathcal{H} . For $A \in \mathcal{B}(\mathcal{H})$, we denote by A^* , $\mathcal{N}(A)$, $\mathcal{R}(A)$, and $\sigma(A)$ the adjoint, the null space, the range and the spectrum of A , respectively. An operator P is called an orthogonal projection if $P = P^* = P^2$. In the following, by a projection we always mean an orthogonal projection. We denote by $\mathcal{P}(\mathcal{H})$ the set of all projections on \mathcal{H} . A self-adjoint operator A is called positive if $\sigma(A) \subseteq [0, +\infty)$. If A is positive, then $A^{\frac{1}{2}}$ denotes the positive square root of A . For any subset \mathcal{S} of $\mathcal{B}(\mathcal{H})$, the commutant of \mathcal{S} is $\mathcal{S}' = \{T \in \mathcal{B}(\mathcal{H}) : TS = ST \text{ for all } S \in \mathcal{S}\}$. We let $W^*(\mathcal{S})$ be the von Neumann algebra generated by \mathcal{S} . It is known that $\{A\}'$ is a von Neumann algebra for any positive operator A . Let \mathcal{D} be a von Neumann algebra, we denote by $\mathcal{U}(\mathcal{D})$ the set of all unitary operators in \mathcal{D} .

Suppose A is a self-adjoint contraction. It is obvious that $\mathcal{N}(A)$, $\mathcal{N}(A - I)$, $\mathcal{N}(A + I)$ and the orthogonal complement \mathcal{H}_0 of the sum of these reduce A . Without any confusion, we denote by I the identity on any Hilbert space. Then

$$\mathcal{H} = \mathcal{N}(A) \oplus \mathcal{N}(A - I) \oplus \mathcal{N}(A + I) \oplus \mathcal{H}_0$$

and

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & -I & 0 \\ 0 & 0 & 0 & A_0 \end{pmatrix},$$

where $A_0 = A|_{\mathcal{H}_0}$. We call A_0 the generic part of A . If $\mathcal{N}(A) = \mathcal{N}(A - I) = \mathcal{N}(A + I) = \{0\}$, then we say that A is in the generic position.

If there exist two projections $P, Q \in \mathcal{P}(H)$ such that $A = P - Q$, then

$$\begin{aligned} \mathcal{N}(A) &= (\mathcal{N}(P) \cap \mathcal{N}(Q)) \oplus (\mathcal{R}(P) \cap \mathcal{R}(Q)), \\ \mathcal{N}(A - I) &= \mathcal{R}(P) \cap \mathcal{N}(Q) \end{aligned}$$

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