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Pairs of orthogonal projections with a fixed difference

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ABSTRACT

Let A be a self-adjoint contraction on a Hilbert space $\mathscr{H}$. In 2014, E. Andruchow proved that if A is a difference of two orthogonal projections, with non-trivial generic part, then there exist infinitely many pairs (P, Q) of orthogonal projections such that $A = P - Q$. In the present paper, we give a sufficient and necessary condition for A to be a difference of a pair of orthogonal projections. Then we give a characterization of all pairs (P, Q) of orthogonal projections such that $A = P - Q$. Moreover, we characterize the von Neumann algebra generated by such pairs (P, Q), and consider the connected components of the set $\{(P, Q) : A = P - Q\}$.

1. Introduction

The subject of pairs of orthogonal projections has been an active area in past decades. During this period, Kato [11], Dixmier [9], Halmos [10] and Davis [8] established many remarkable results. Recently, several related problems are studied by a number of au-

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thors [1,3–7,12]. In [8], C. Davis characterized the self-adjoint operators which are a difference of two orthogonal projections. Afterwards, E. Andruchow [2] proved that if A is a difference of two orthogonal projections, with non-trivial generic part, then there exist infinitely many pairs (P, Q) of orthogonal projections such that $A = P - Q$. It is therefore natural to ask for a description of all pairs (P, Q) of orthogonal projections satisfying $A = P - Q$. On the other hand, from the perspective of algebra, we need to know what is the von Neumann algebra generated by such pairs (P, Q) of orthogonal projections. We put $A_P = \{(P, Q) : A = P - Q\}$. The objective of this paper is twofold. The first one is to characterize A_P, the other is to get the von Neumann algebra generated by all pairs $(P, Q) \in A_P$ of orthogonal projections and its commutant.

Throughout this paper, let $\mathscr{H}$ be a Hilbert space and $\mathscr{B}(\mathscr{H})$ the algebra of all bounded linear operators on $\mathscr{H}$. For $A \in \mathscr{B}(\mathscr{H})$, we denote by A^*, $\mathscr{N}(A)$, $\mathscr{R}(A)$, and $\sigma(A)$ the adjoint, the null space, the range and the spectrum of A, respectively. An operator P is called an orthogonal projection if $P = P^* = P^2$. In the following, by a projection we always mean an orthogonal projection. We denote by $\mathscr{P}(\mathscr{H})$ the set of all projections on $\mathscr{H}$. A self-adjoint operator A is called positive if $\sigma(A) \subseteq [0, +\infty)$. If A is positive, then $A^{\frac{1}{2}}$ denotes the positive square root of A. For any subset $\mathscr{S}$ of $\mathscr{B}(\mathscr{H})$, the commutant of $\mathscr{S}$ is $\mathscr{S}' = \{T \in \mathscr{B}(\mathscr{H}) : TS = ST \text{ for all } S \in \mathscr{S}\}$. We let $W^*(\mathscr{S})$ be the von Neumann algebra generated by $\mathscr{S}$. It is known that $\{A\}'$ is a von Neumann algebra for any positive operator A. Let $\mathscr{D}$ be a von Neumann algebra, we denote by $\mathscr{U}(\mathscr{D})$ the set of all unitary operators in $\mathscr{D}$.

Suppose A is a self-adjoint contraction. It is obvious that $\mathscr{N}(A)$, $\mathscr{N}(A-I)$, $\mathscr{N}(A+I)$ and the orthogonal complement $\mathscr{H}_0$ of the sum of these reduce A. Without any confusion, we denote by I the identity on any Hilbert space. Then

$$\mathscr{H} = \mathscr{N}(A) \oplus \mathscr{N}(A - I) \oplus \mathscr{N}(A + I) \oplus \mathscr{H}_0$$

and

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & -I & 0 \\ 0 & 0 & 0 & A_0 \end{pmatrix},$$

where $A_0 = A|_{\mathscr{H}_0}$. We call A_0 the generic part of A. If $\mathscr{N}(A) = \mathscr{N}(A - I) = \mathscr{N}(A + I) = \{0\}$, then we say that A is in the generic position.

If there exist two projections $P, Q \in \mathscr{P}(H)$ such that $A = P - Q$, then

$$\mathscr{N}(A) = (\mathscr{N}(P) \cap \mathscr{N}(Q)) \oplus (\mathscr{R}(P) \cap \mathscr{R}(Q)),$$

$$\mathscr{N}(A - I) = \mathscr{R}(P) \cap \mathscr{N}(Q)$$