Linear Algebra and its Applications 485 (2015) 1–20

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Generic properties and a criterion of an operator norm ☆

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A R T I C L E   I N F O

Article history:
Received 28 October 2014
Accepted 17 July 2015
Available online 25 July 2015
Submitted by V. Mehrmann

MSC:
15A60
15A69

Keywords:
Operator norm
Injective norm
Subdifferential
Gradient

A B S T R A C T

In this paper, we carefully examine the structure of the gradient of an operator norm on a finite-dimensional matrix space. In particular, we derive concise and useful representations for an operator norm and its subgradient, which refine existing results in this area of study. We further use the derived representations to formulate and prove a criterion of an operator norm, the first of its kind, to the best of our knowledge. It essentially states that *a matrix norm is an operator norm if and only if the set of its gradients is the set of the outer products of vectors from each pair of the Cartesian product of two vector sets*. We also provide several handy tests, based on this criterion, which in certain cases help to determine whether a matrix norm is an operator norm or not. In addition, we generalize our theoretical developments to higher dimensions, i.e. for injective norms on tensor spaces.

☆ This work was supported by Russian Scientific Foundation grant 14-11-00806.

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http://dx.doi.org/10.1016/j.laa.2015.07.020
0024-3795/

1. Introduction

Operator norms are a special family of norms on matrix spaces which reflect the properties of matrices as bounded operators between normed vector spaces. An operator norm, generally defined by the equation

$$\|A\| = \max_{x \neq 0} \frac{\|Ax\|_*}{\|x\|_{**}},$$

can be viewed as a measure of the "maximum stretch" performed by the operator A. It is also equal to the minimum value C such that the inequality

$$\|Ax\|_* \leq C\|x\|_{**}$$

holds true for all vectors x of the corresponding vector space.

These properties evoke great interest to operator norms in various applications, e.g. stability analysis of linear dynamical systems [1], robust optimization [2], certain NP-hard problems (the Max-Cut problem and the Stones problem) [2], Khot's Unique Games Conjecture [3], etc. Majorization and approximate computation of operator norms play an important role in these applications. It has been shown that approximate computation of many operator norms, for instance, norms on $\mathcal{L}(\ell_p, \ell_r)$ for almost all p and r such that $1 \leq p, r \leq \infty$, is NP-hard [2,4].

In this paper, we focus on the question "is a particular matrix norm an operator norm or not?" This question has been answered in the negative for the Frobenius norm [5], as well as for all other unitarily invariant norms except for the spectral norm. The argument is quite simple: since all operator norms satisfy the equation

$$\|xy^T\| = \|x\|_* \|y\|_{**}^D,$$

and all unitarily invariant norms satisfy the equation

$$\|xy^T\| = \text{const} \cdot \|x\|_2 \|y\|_2,$$

the only possible unitarily invariant operator norm is

$$\|A\| = \text{const} \cdot \max_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2} = \text{const} \cdot \|A\|_2.$$

The argument for the Frobenius norm heavily depends on the fact that *we know exactly* that it is not proportional to the spectral norm. It means that we cannot apply this argument directly to another norm of our interest, e.g. the p-norm of vectorized matrices

$$\|A\|_{\text{vec}(p)} = \left(\sum_{i,j} |a_{ij}|^p \right)^{1/p}, \quad p \geq 1.$$