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Graph functions maximized on a path

Celso Marques da Silva Jr. [a,*,1], Vladimir Nikiforov [b]

[a] *PEP-COPPE, Universidade Federal do Rio de Janeiro and Centro Federal de Educação Tecnológica Celso Suckow da Fonseca, Rio de Janeiro, Brazil*
[b] *Department of Mathematical Sciences, University of Memphis, Memphis, TN 38152, USA*

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A B S T R A C T

Given a connected graph G of order n and a nonnegative symmetric matrix $A = [a_{i,j}]$ of order n, define the function $F_A(G)$ as

$$F_A(G) = \sum_{1 \le i < j \le n} d_G(i,j)\, a_{i,j},$$

where $d_G(i,j)$ denotes the distance between the vertices i and j in G.
In this note it is shown that $F_A(G) \le F_A(P)$ for some path of order n. Moreover, if each row of A has at most one zero off-diagonal entry, then $F_A(G) < F_A(P)$ for some path of order n, unless G itself is a path.
In particular, this result implies two conjectures of Aouchiche and Hansen:

– the spectral radius of the distance Laplacian of a connected graph G of order n is maximal if and only if G is a path;
– the spectral radius of the distance signless Laplacian of a connected graph G of order n is maximal if and only if G is a path.

* Corresponding author.
E-mail addresses: celsomjr@gmail.com (C. Marques da Silva), vnikifrv@memphis.edu (V. Nikiforov).
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1. Introduction and main results

The aim of the present note is to give a general approach to problems like the following conjectures of Aouchiche and Hansen [1,2]:

Conjecture 1. *The largest eigenvalue of the distance Laplacian of a connected graph G of order n is maximal if and only if G is a path.*

Conjecture 2. *The largest eigenvalue of the distance signless Laplacian of a connected graph G of order n is maximal if and only if G is a path.*

First, let us introduce some notation and recall a few definitions. We write $\lambda(A)$ for the largest eigenvalue of a symmetric matrix A. Given a connected graph G, let $D(G)$ be the distance matrix of G, and let $T(G)$ be the diagonal matrix of the rowsums of $D(G)$. The matrix $D^L(G) = T(G) - D(G)$ is called the *distance Laplacian* of G, and the matrix $D^Q(G) = T(G) + D(G)$ is called the *distance signless Laplacian* of G. The matrices $D^L(G)$ and $D^Q(G)$ have been introduced by Aouchiche and Hansen and have been intensively studied recently, see, e.g., [1–3,5,7,12].

Very recently, Lin and Lu [5] succeeded to prove Conjecture 2, but Conjecture 1 seems a bit more difficult and still holds. Furthermore, Conjectures 1 and 2 suggest a similar problem for the distance matrix itself. As it turns out such problem has been partially solved a while ago by Ruzieh and Powers [9], who showed that the largest eigenvalue of the distance matrix of a connected graph G of order n is maximal if G is a path. The complete solution, however, was given more recently by Stevanović and Ilić [10].

Theorem 3. *(See [9,10].) The largest eigenvalue of the distance matrix of a connected graph G of order n is maximal if and only if G is a path.*

These result are believed to belong to spectral graph theory, and their proofs involve nonnegligible amount of calculations. Our goal is to show that all these result stem from a much more general assertion that has nothing to do with eigenvalues. To this end, we shall introduce a fairly general graph function and shall study its maxima.

1.1. The function $F_A(G)$ and its maxima

Let G be a connected graph of order n. Write $d_G(i,j)$ for the distance between the vertices i and j in G, and let $A = [a_{i,j}]$ be a nonnegative symmetric matrix of order n. Define the function $F_A(G)$ as