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# Inverse of the distance matrix of a cycle–clique graph

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ABSTRACT

A connected graph $G$, all of whose blocks are cycles or cliques, is called a cycle–clique graph. Let $D$ be the distance matrix of $G$. By a theorem of Graham et al., we have $det(D) \neq 0$ if all cycle blocks have odd vertices. In this paper we give the formula for the inverse of $D$.



## 1. Introduction

All graphs in this paper are connected and simple. In a graph $G$, the distance $d(u, v)$ between two vertices $u$, $v$ is the length of the shortest path between them. The *distance matrix* $D(G)$ of a graph $G$ is a matrix with $(u, v)$-entry $d(u, v)$. Let $A$ be an $n \times n$ matrix.

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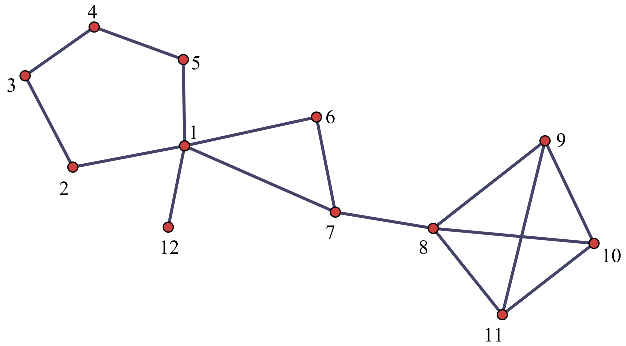


**Fig. 1.** An example of cycle–clique graph.

Recall that the cofactor $c_{i,j}$ is defined as $(-1)^{i+j}$ times the determinant of the submatrix obtained by deleting row $i$ and column $j$ of $A$. Let $Cof(A) = \sum_{i,j} c_{i,j}$ be the sum of all cofactors of $A$. Graham, Hoffman and Hosoya [5] proved a very attractive theorem about the determinant of the distance matrix $D(G)$ of a connected graph $G$ as a function of the distance matrix of its *blocks*:

**Theorem 1.1.** *Let $G$ be a graph with blocks $G_1, G_2, \cdots, G_r$. Then*

$$Cof(D(G)) = \prod_{i=1}^{r} Cof(D(G_i))$$

$$\det(D(G)) = \sum_{i=1}^{r} \det(D(G_i)) \prod_{j \neq i} Cof(D(G_j)).$$

A graph $G$ is called a *cycle–clique graph* if all of its blocks are cycles or cliques. For example, unicyclic graphs (connected graphs with the same number of vertices and edges) and cactus graphs (connected graphs whose blocks are cycles or edges) are cycle–clique graphs. The graph depicted in Fig. 1 is a cycle–clique graph with five blocks $C_5$, $K_2$, $C_3$, $K_2$, $K_4$.

From Theorem 1.1, we can give a formula for the determinant of the distance matrix $D(G)$ of a cycle–clique graph $G$ in terms of the sizes of its blocks and the number of all vertices. By Lemma 2.4, we see that $\det D(G) \neq 0$ if and only if all cycle blocks of $G$ have odd number of vertices. Hence, we are interested in finding the inverse $D(G)^{-1}$ of $D(G)$.

For the case when all blocks of $G$ are $K_2$ (i.e. $G$ is a tree) it is known [1,6] that $D(G)^{-1} = -\frac{L(G)}{2} + \frac{1}{2(n-1)}\tau\tau^T$, where $L(G)$ is the Laplacian matrix of $G$ and $\tau$ is the $n \times 1$ column vector with $\tau(v) = 2 - deg(v)$. Similarly, it is known that when all blocks are cliques [3] we have $D(G)^{-1} = -\hat{L} + \frac{1}{\lambda_G}\beta\beta^T$ where $\hat{L}$ is a Laplacian-like matrix of $G$ and $\beta$ is a suitable column vector (see [3] for details). Thus $D^{-1}$ is a constant times $L$ plus a multiple of a rank one matrix. Some similar results can be found in [2,4,7,8]. We