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On lattices of closed subgroups in the group of infinite triangular matrices over a field

Agnieszka Bier

Institute of Mathematics, Silesian University of Technology, ul. Kaszubska 23, 44-100 Gliwice, Poland

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ABSTRACT

We investigate a special type of closed subgroups of the topological group $\mathrm{UT}(\infty, K)$ of infinite-dimensional unitriangular matrices over a field K ($|K| > 2$), considered with the natural inverse limit topology. Namely, we generalize the concept of partition subgroups introduced in [23] and define partition subgroups in $\mathrm{UT}(\infty, K)$. We show that they are all closed and discuss the problem of their invariance to various group homomorphisms. We prove that a characteristic subgroup of $\mathrm{UT}(\infty, K)$ is necessarily a partition subgroup and characterize the lattices of characteristic and fully characteristic subgroups in $\mathrm{UT}(\infty, K)$. We conclude with some implications of the given characterization on verbal structure of $\mathrm{UT}(\infty, K)$ and $\mathrm{T}(\infty, K)$ and use some topological properties to discuss the problem of the width of verbal subgroups in groups defined over a finite field K.

1. Introduction

Let K be a field such that $|K| > 2$. By $\mathrm{T}(n, K)$ we denote the group of all invertible upper triangular matrices of size $n \times n$ over the field K. Further, by $\mathrm{UT}(n, K)$ we denote

E-mail address: agnieszka.bier@polsl.pl.

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the subgroup of $\mathrm{T}(n, K)$ consisting of all unitriangular matrices (i.e. the triangular matrices having all diagonal entries equal to 1), and by $\mathrm{D}(n, K)$ we denote the subgroup of $\mathrm{T}(n, K)$ consisting of all diagonal matrices with nonzero diagonal entries. For $i > j$ the group $\mathrm{T}(i, K)$ (and so $\mathrm{UT}(i, K)$, and $\mathrm{D}(i, K)$) may be mapped onto $\mathrm{T}(j, K)$ (respectively $\mathrm{UT}(j, K)$ and $\mathrm{D}(j, K)$) using the projection π_{ij} (or its restrictions $\pi_{ij}|_{\mathrm{UT}(i,K)}$ and $\pi_{ij}|_{\mathrm{D}(i,K)}$), which deletes the last $(i - j)$ rows and the last $(i - j)$ columns of the matrix. The limits of the obtained inverse spectra $(\mathrm{T}(i, K), \pi_{i,i-1})$, $\big(\mathrm{UT}(i, K), \pi_{i,i-1}|_{\mathrm{UT}(i,K)}\big)$ and $\big(\mathrm{D}(i, K), \pi_{i,i-1}|_{\mathrm{D}(i,K)}\big)$ will be denoted by $\mathrm{T}(\infty, K)$, $\mathrm{UT}(\infty, K)$ and $\mathrm{D}(\infty, K)$ respectively, and called the groups of infinite triangular, infinite unitriangular and infinite diagonal matrices. The elements of $\mathrm{T}(\infty, K)$, $\mathrm{UT}(\infty, K)$ and $\mathrm{D}(\infty, K)$ are the matrices with entries indexed by the set $\mathbb{N} \times \mathbb{N}$. The group $\mathrm{UT}(\infty, K)$ contains as a subgroup the stable group $\mathrm{UT}_f(\infty, K)$ of all finitary infinite matrices, which may be constructed as a direct limit of groups $\mathrm{UT}(n, K)$, $n \in \mathbb{N}$, with natural embeddings. Similarly, the direct limits of triangular and diagonal matrix groups will be denoted by $\mathrm{T}_f(\infty, K)$ and $\mathrm{D}_f(\infty, K)$, respectively.

In the past few years, the groups of infinite matrices have drawn attention of many researchers [5–7,16,17]. Among others one finds results on various aspects of groups $\mathrm{T}(\infty, K)$ and $\mathrm{UT}(\infty, K)$, like those concerning their subgroup structure, their automorphisms, or solvability of special types of equations [3,4,18–20]. Being inverse limits, the groups $\mathrm{T}(\infty, K)$ and $\mathrm{UT}(\infty, K)$ may be considered in a natural way as topological groups, and in particular – profinite groups, if K is finite (for more information on profinite groups see [13] and [14]). In the latter case, the topological properties of $\mathrm{T}(\infty, K)$ and $\mathrm{UT}(\infty, K)$ turn out to be interesting both as a self-contained study and as a tool for investigations of the verbal structure in these groups [15]. This was the motivation of the research presented within this paper.

Throughout the paper all finitely dimensional matrices will be denoted with lowercase letters, while for the infinite matrices we will use the uppercase letters. For every matrix $a \in \mathrm{UT}(n, K)$ (or $A \in \mathrm{UT}(\infty, K)$) and $m \leq n$ by $a[m]$ (and $A[m]$, respectively) we denote the top-left block of size $m \times m$ of matrix a (or A). The identity matrices in the groups $\mathrm{UT}(n, K)$ and $\mathrm{UT}(\infty, K)$ will be denoted by e_n and E. Every finitely dimensional unitriangular matrix $a \in \mathrm{UT}(n, K)$ may be written as a sum:

$$a = e_n + \sum_{1 \leq i < j \leq n} a_{ij} e_{ij},$$

where e_{ij} denotes elementary matrix of size equal to the size of a, which has 1 in the place (i, j) and zeros elsewhere (infinite elementary matrices will be denoted by E_{ij}). Every matrix $A \in \mathrm{UT}_f(\infty, K)$ (or in $\mathrm{T}_f(\infty, K)$ or $\mathrm{D}_f(\infty, K)$) differs from E only in a finite block $A[n]$ for some n.

In groups $\mathrm{UT}(n, K)$, $\mathrm{UT}(\infty, K)$ and $\mathrm{UT}_f(\infty, K)$ we distinguish the respective subgroups $\mathrm{UT}(n, m, K)$, $\mathrm{UT}(\infty, m, K)$ and $\mathrm{UT}_f(\infty, m, K)$, which consist of all those matrices, whose all entries on the first m superdiagonals are zeros. It is well known (see e.g. [8]) that the series of subgroups