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The Kalman–Yakubovich–Popov inequality for differential-algebraic systems

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ABSTRACT

In this paper we revisit the Kalman–Yakubovich–Popov lemma for differential-algebraic control systems. This lemma relates the positive semi-definiteness of the Popov function on the imaginary axis to the solvability of a linear matrix inequality on a certain subspace. Further emphasis is placed on the Lur'e equation, whose solution set consists, loosely speaking, of the rank-minimizing solutions of the Kalman–Yakubovich–Popov inequality. We show that there is a correspondence between the solution set of the Lur'e equation and the deflating subspaces of certain even matrix pencils. Finally, we show that under certain conditions the Lur'e equation admits stabilizing, anti-stabilizing, and extremal solutions. We note that, for our results, we neither assume impulse controllability nor make any assumptions on the index of the system.

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1. Introduction

In this work we consider differential-algebraic control systems (or descriptor systems) of the form

$$E\dot{x}(t) = Ax(t) + Bu(t), \tag{1.1}$$

where $E, A \in \mathbb{K}^{n\times n}$ such that the pencil $sE - A \in \mathbb{K}[s]^{n\times n}$ is regular (see Definition 2.1 (a)) and $B \in \mathbb{K}^{n\times m}$ (for the notation of this article we refer to the end of this introductory section). The set of these systems is denoted by $\Sigma_{n,m}(\mathbb{K})$ and we write $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$. The function $u : \mathbb{R} \to \mathbb{K}^m$ is called *input* of the system; we call $x(t) \in \mathbb{K}^n$ the *(generalized) state* of $[E, A, B]$ at time $t \in \mathbb{R}$. The set of solution trajectories $(x, u) : \mathbb{R} \to \mathbb{K}^n \times \mathbb{K}^m$ induces the *behavior* of (1.1):

$$\begin{aligned}\mathfrak{B}_{[E,A,B]} := \big\{(x,u) \in \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R},\mathbb{K}^n) \times \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R},\mathbb{K}^m) : E\dot{x} \in \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R},\mathbb{K}^n) \\ \text{and } (x,u) \text{ solves (1.1) for almost all } t \in \mathbb{R}\big\}.\end{aligned}$$

The main algebraic concept for our considerations is the *Popov function*, which is defined by

$$\Phi(s) = \begin{bmatrix} (-\bar{s}E - A)^{-1}B \\ I_m \end{bmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} (sE - A)^{-1}B \\ I_m \end{bmatrix} \in \mathbb{K}(s)^{m\times m},$$

where $Q = Q^* \in \mathbb{K}^{n\times n}$, $S \in \mathbb{K}^{n\times m}$, and $R = R^* \in \mathbb{K}^{m\times m}$ are given matrices. Note that $\Phi(\mathrm{i}\omega)$ is Hermitian for all $\omega \in \mathbb{R}$ with $\det(\mathrm{i}\omega E - A) \neq 0$. In particular, we are going to study algebraic conditions for the pointwise positive semi-definiteness of $\Phi(\mathrm{i}\,\cdot\,) : \{\omega \in \mathbb{R} : \det(\mathrm{i}\omega E - A) \neq 0\} \to \mathbb{C}^{m\times m}$. This property is strongly related to the feasibility of the linear-quadratic optimal control problem in which the cost functional is formed by the matrix $\left[\begin{smallmatrix} Q & S \\ S^* & R \end{smallmatrix}\right]$, see, e.g., [46].

In the case of ordinary differential equations (that is, $E = I_n$), the pointwise positive semi-definiteness of $\Phi(\mathrm{i}\cdot)$ can be assessed by the famous Kalman–Yakubovich–Popov lemma, see, e.g., [1,22,37,38,49] and the references therein. More precisely, under certain assumptions related to controllability, this property is equivalent to the solvability of the Kalman–Yakubovich–Popov (KYP) inequality, namely there exists a $P \in \mathbb{K}^{n\times n}$ such that

$$\begin{bmatrix} A^*P + PA + Q & PB + S \\ B^*P + S^* & R \end{bmatrix} \geq 0, \quad P = P^*. \tag{1.2}$$

There are several attempts to generalize this lemma to differential-algebraic equations: For instance, in [33], the case where $sE - A$ is regular and of index at most one has been treated. In [8,9,48] the KYP inequality has been considered for the even more general class of *linear time-invariant behaviors*. In these articles, behavioral controllability has