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Relative numerical ranges

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Relying on the ideas of Stampfli [14] and Magajna [12] we introduce, for operators S and T on a separable complex Hilbert space, a new notion called the numerical range of S relative to T at $r \in \sigma(|T|)$. Some properties of these numerical ranges are proved. In particular, it is shown that the relative numerical ranges are non-empty convex subsets of the closure of the ordinary numerical range of S. We show that the position of zero with respect to the relative numerical range of S relative to T at $\|T\|$ gives an information about the distance between the involved operators. This result has many interesting corollaries. For instance, one can characterize those complex numbers which are in the closure of the numerical range of S but are not in the spectrum of S.

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1. Introduction

Let $\mathscr{H}$ be a separable complex Hilbert space. We denote by $\mathscr{S}_{\mathscr{H}}$ the unit sphere of $\mathscr{H}$ and by $\mathcal{B}(\mathscr{H})$ the Banach algebra of all bounded linear operators on $\mathscr{H}$. The numerical range of $S \in \mathcal{B}(\mathscr{H})$ is $W(S) = \{\langle Sx, x\rangle;\ x \in \mathscr{S}_{\mathscr{H}}\}$. It is well-known that $W(S)$ is a non-empty convex subset of the disc $\{z \in \mathbb{C};\ |z| \le \|S\|\}$. The reader is referred to [8–10] for details about the numerical ranges. Many important properties of an operator are encoded in its numerical range. For instance, $\sigma(S)$, the spectrum of S, is a subset of $\overline{W(S)}$, the closure of the numerical range. In this paper, we are interested in some parts of $\overline{W(S)}$ which are specified by an operator $T \in \mathcal{B}(\mathscr{H})$. We call them relative numerical ranges. They carry some useful information about the relation between S and T. We would like to point out that our study partially relies on the ideas of Stampfli [14] and Magajna [12].

The article is divided into two parts. In the first part, Section 2, we give a motivation for our definition of the relative numerical ranges and then we explore their properties. Among them the most important is convexity (Theorem 2.6). In the second part of the paper, Section 3, we show that the position of zero with respect to a relative numerical range gives an information about the distance between the involved operators. The main result of the section is Theorem 3.3 which extends a result proved by Stampfli [14] and therefore its proof is a modification of Stampfli's ideas, see also [6,12]. Theorem 3.3 has several interesting corollaries, one of them is a characterization of the set $\overline{W(S)} \setminus \sigma(S)$ for any $S \in \mathcal{B}(\mathscr{H})$.

2. Definition of the relative numerical ranges and their properties

To motivate our definition of the relative numerical ranges, we begin with a basic property of the ordinary numerical range: the numerical range of a compression of S to a closed subspace of $\mathscr{H}$ is contained in $W(S)$. More precisely, if $\mathscr{K}$ is a closed subspace of $\mathscr{H}$ and P is the orthogonal projection onto $\mathscr{K}$, then $W(PS|_{\mathscr{K}}) \subseteq W(S)$. The following lemma gives a description of $\overline{W(PS|_{\mathscr{K}})}$ which is the key idea in our definition of the relative numerical ranges.

Lemma 2.1. *Let $\mathscr{K} \neq \{0\}$ be a closed subspace of $\mathscr{H}$ and P be the orthogonal projection onto $\mathscr{K}$. Then, for every $S \in \mathcal{B}(\mathscr{H})$,*

$$\overline{W(PS|_{\mathscr{K}})} = \{\lambda \in \mathbb{C};\ \exists\, (x_n)_{n=1}^{\infty} \subseteq \mathscr{S}_{\mathscr{H}} : \lim_{n\to\infty} \|Px_n\| = \|P\| \quad \textit{and} \quad \lim_{n\to\infty} \langle Sx_n, x_n\rangle = \lambda\}. \tag{2.1}$$

Proof. Assume that $\lambda \in \overline{W(PS|_{\mathscr{K}})}$. Then there exists a sequence $(\lambda_n)_{n=1}^{\infty} \subseteq W(PS|_{\mathscr{K}})$ such that $\lambda = \lim_{n\to\infty} \lambda_n$. Hence there is a sequence $(x_n)_{n=1}^{\infty} \subseteq \mathscr{S}_{\mathscr{K}}$ such that $\lambda_n = \langle PS|_{\mathscr{K}} x_n, x_n\rangle$ for every $n \in \mathbb{N}$. Since $Px_n = x_n$ and therefore $\|Px_n\| = 1 = \|P\|$,