Linear Algebra and its Applications 485 (2015) 222–249

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

On a type of commutative algebras ☆

A.L. Agore [a,b,*], G. Militaru [c]

[a] Faculty of Engineering, Vrije Universiteit Brussel, Pleinlaan 2, B-1050 Brussels, Belgium
[b] Department of Applied Mathematics, Bucharest University of Economic Studies, Piata Romana 6, RO-010374 Bucharest 1, Romania
[c] Faculty of Mathematics and Computer Science, University of Bucharest, Str. Academiei 14, RO-010014 Bucharest 1, Romania

ARTICLE INFO

Article history:
Received 23 March 2015
Accepted 29 July 2015
Available online 7 August 2015
Submitted by P. Semrl

MSC:
17C10
17C55

Keywords:
Jacobi–Jordan algebras
Crossed products
The extension problem
Non-abelian cohomology

ABSTRACT

We introduce some basic concepts for Jacobi–Jordan algebras such as: representations, crossed products or Frobenius/metabelian/co-flag objects. A new family of solutions for the quantum Yang–Baxter equation is constructed arising from any 3-step nilpotent Jacobi–Jordan algebra. Crossed products are used to construct the classifying object for the extension problem in its global form. For a given Jacobi–Jordan algebra A and a given vector space V of dimension $\mathfrak{c}$, a global non-abelian cohomological object $\mathbb{GH}^2(A, V)$ is constructed: it classifies, from the view point of the extension problem, all Jacobi–Jordan algebras that have a surjective algebra map on A with kernel of dimension $\mathfrak{c}$. The object $\mathbb{GH}^2(A, k)$ responsible for the classification of co-flag algebras is computed, all $1 + \dim(A)$ dimensional Jacobi–Jordan algebras that have an algebra surjective map on A are classified and the automorphism groups of these algebras is determined. Several examples involving special sets of matrices and symmetric bilinear forms as well as equivalence

☆ A.L. Agore is Postoctoral Fellow of the Fund for Scientific Research – Flanders (Belgium) (F.W.O. Vlaanderen). This work was supported by a grant of the Romanian National Authority for Scientific Research, CNCS-UEFISCDI, grant no. 88/05.10.2011.
* Corresponding author.
E-mail addresses: ana.agore@vub.ac.be, ana.agore@gmail.com (A.L. Agore), gigel.militaru@fmi.unibuc.ro, gigel.militaru@gmail.com (G. Militaru).

http://dx.doi.org/10.1016/j.laa.2015.07.035

relations between them (generalizing the isometry relation) are provided.

0. Introduction

Jacobi–Jordan algebras (JJ algebras for short) were recently introduced in [5] as vector spaces A over a field k, equipped with a bilinear map $\cdot : A \times A \to A$ satisfying the Jacobi identity and instead of the skew-symmetry condition valid for Lie algebras we impose commutativity $x \cdot y = y \cdot x$, for all x, $y \in A$. These type of algebras appeared already in relation with Bernstein algebras in 1987 [24]. One crucial remark is that JJ algebras are examples of the more popular and well-referenced Jordan algebras [13] introduced in order to achieve an axiomatization for the algebra of observables in quantum mechanics. In [5] the authors achieved the classification of these algebras up to dimension 6 over an algebraically closed field of characteristic different from 2 and 3. Our purpose is to introduce and develop some basic concepts for JJ algebras which might eventually lead to an interesting theory. As it was explained in [5] and as it will be obvious from this paper as well, JJ algebras are objects fundamentally different from both associative and Lie algebras even if their definition differs from the latter only modulo a sign. We aim to prove that there exists a rich and very interesting theory behind the Jacobi–Jordan algebras which deserves to be developed further mainly for three reasons. The first reason is a theoretical one: JJ algebras are objects of study in their own right as objects living at the interplay between the intensively studied Lie algebras and respectively associative algebras. In this context it is a challenge to introduce the JJ algebra counterparts of concepts already defined in the theory of Lie (resp. associative) algebras and to see which of the results valid in the fields mentioned above are also true for JJ algebras. The second reason comes from the observation that JJ algebras are a special class of Jordan algebras which turned out to play a fundamental role not only in quantum mechanics but also in differential geometry, algebraic geometry or functional analysis [18] – from this perspective they deserve a detailed study. Finally, the third reason is given by the connection which we will highlight in Section 1 between JJ algebras and the celebrated quantum Yang–Baxter equation from theoretical physics. A central open problem in this context is to construct new families of solutions: here we take a first step towards it by associating to any 3-step nilpotent JJ algebra a new family of solutions for the quantum Yang–Baxter equation. Therefore, constructing such algebras becomes a matter of interest.

The paper is organized as follows. The first section fixes notations and conventions used throughout and introduces some basic concepts in the context of JJ algebras such as modules, representations or Frobenius objects. In order to find the right axiom for defining modules over a JJ algebra (Definition 1.4) we use the classical trick used for objects