Linear Algebra and its Applications 485 (2015) 250–266

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

On the spectrum in max algebra

Vladimir Müller [a], Aljoša Peperko [b,c,*]

[a] Institute of Mathematics, Czech Academy of Sciences, Žitna 25, 115 67 Prague, Czech Republic

[b] Faculty of Mechanical Engineering, University of Ljubljana, Aškerčeva 6, SI-1000 Ljubljana, Slovenia

[c] Institute of Mathematics, Physics and Mechanics, Jadranska 19, SI-1000 Ljubljana, Slovenia

A R T I C L E I N F O

Article history:
Received 14 November 2014
Accepted 10 July 2015
Available online 7 August 2015
Submitted by P. Šemrl

MSC:
15A18
15A80
15A60
15B48

Keywords:
Non-negative matrices
Max algebra
Eigenvalues
Distinguished eigenvalues
Spectral mapping theorem in max algebra
Power series in max algebra
Continuity

A B S T R A C T

We give new proofs of several fundamental results of spectral theory in max algebra. This includes the description of the spectrum in max algebra of a given non-negative matrix via local spectral radii, the spectral theorem and the spectral mapping theorem in max algebra. The latter result is also generalized to the setting of power series in max algebra by applying certain continuity properties of the spectrum in max algebra. Our methods enable us to obtain some related results for the usual spectrum of complex matrices and distinguished spectrum for non-negative matrices.

* Corresponding author at: Faculty of Mechanical Engineering, University of Ljubljana, Aškerčeva 6, SI-1000 Ljubljana, Slovenia.

E-mail addresses: muller@math.cas.cz (V. Müller), aljosa.peperko@fmf.uni-lj.si (A. Peperko).

http://dx.doi.org/10.1016/j.laa.2015.07.013

1. Introduction

The algebraic system max algebra and its isomorphic versions (max-plus algebra, tropical algebra) provide an attractive way of describing a class of non-linear problems appearing for instance in manufacturing and transportation scheduling, information technology, discrete event-dynamic systems, combinatorial optimization, mathematical physics, DNA analysis, ... (see e.g. [11,6,17,7,10,21,30,25] and the references cited there). Max algebra's usefulness arises from a fact that these non-linear problems become linear when described in the max algebra language. Moreover, recently max algebra techniques were used to solve certain linear algebra problems (see e.g. [15,28]). In particular, tropical polynomial methods improved the accuracy of the numerical computation of the eigenvalues of a matrix polynomial (see e.g. [1,2,18,19,4] and the references cited there).

The max algebra consists of the set of non-negative numbers with sum $a \oplus b = \max\{a, b\}$ and the standard product ab, where $a, b \geq 0$. A matrix $A = [A_{ij}]_{i,j=1}^n$ is non-negative, if $A_{ij} \geq 0$ for all $i, j \in \{1, 2, \ldots, n\}$. Let $\mathbb{R}^{n\times n}$ ($\mathbb{C}^{n\times n}$) be the set of all $n \times n$ real (complex) matrices and $\mathbb{R}_+^{n\times n}$ the set of all $n \times n$ non-negative matrices. The operations between matrices and vectors in the max algebra are defined by analogy with the usual linear algebra. The product of non-negative matrices A and B in the max algebra is denoted by $A \otimes B$, where $(A \otimes B)_{ij} = \max_{k=1,\ldots,n} A_{ik}B_{kj}$ and the sum $A \oplus B$ in the max algebra is defined by $(A \oplus B)_{ij} = \max\{A_{ij}, B_{ij}\}$. The notation $A_\otimes^2$ means $A \otimes A$, and $A_\otimes^k$ denotes the k-th max power of A. If $x = (x_i)_{i=1,\ldots,n}$ is a non-negative vector, then the notation $A \otimes x$ means $(A \otimes x)_i = \max_{j=1,\ldots,n} A_{ij}x_j$. The usual associative and distributive laws hold in this algebra.

The role of the spectral radius of $A \in \mathbb{R}_+^{n\times n}$ in max algebra is played by the maximum cycle geometric mean $r_\otimes(A)$, which is defined by

$$r_\otimes(A) = \max\Big\{(A_{i_1 i_k} \cdots A_{i_3 i_2} A_{i_2 i_1})^{1/k} : k \in \mathbb{N} \text{ and } i_1, \ldots, i_k \in \{1, \ldots, n\}\Big\} \tag{1}$$

and equal to

$$r_\otimes(A) = \max\Big\{(A_{i_1 i_k} \cdots A_{i_3 i_2} A_{i_2 i_1})^{1/k} : k \leq n \text{ and } i_1, \ldots, i_k \in \{1, \ldots, n\} \text{ mutually distinct}\Big\}.$$

A digraph $\mathcal{G}(A) = (N(A), E(A))$ associated to $A \in \mathbb{R}_+^{n\times n}$ is defined by setting $N(A) = \{1, \ldots, n\}$ and letting $(i, j) \in E(A)$ whenever $A_{ij} > 0$. When this digraph contains at least one cycle, one distinguishes critical cycles, where the maximum in (1) is attained. A graph with just one node and no edges will be called trivial. A bit unusually, but in consistency with [11,12,22,13], a matrix $A \in \mathbb{R}_+^{n\times n}$ is called irreducible if $\mathcal{G}(A)$ is trivial (A is 1×1 zero matrix) or strongly connected (for each $i, j \in N(A)$ there is a path in $\mathcal{G}(A)$ that starts in i and ends in j).