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Differential quiver algebras and tree modules

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ABSTRACT

In this paper we extend a theorem of Ringel on the existence of tree bases for exceptional modules [1] to the context of quiver algebras with differential, using reduction techniques. We give some examples which show that the reduction functors provide efficient and implementable algorithms for the analysis and construction of tree presentations, including minimal projective presentations, representations of posets and Kronecker modules.

0. Introduction

Let k be an arbitrary field. Ringel has shown [1] that every exceptional module of a finite quiver can be expressed, in a minimal way, by 0–1-matrices (its coefficient quiver is a tree). In this paper we generalize this result to a special kind of differential tensor

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algebras (called nested by Bautista, Salmerón and Zuazua in [2], compare with differential biquivers defined by Crawley-Boevey [3]), using classical reduction techniques (the background of these subjects is reviewed in Sections 1 and 2). This route was initially taken by Crawley-Boevey when considering tree representations for quivers of finite representation type [3], and was in general suggested by the referee of [1].

Theorem. *Any exceptional representation of a differential quiver algebra is a tree module.*

An important feature of the reduction functors is that they produce bases which preserve nice properties from old bases (tree properties, 0–1-coefficients and in some cases independence over the field as in Problem 9 of [1]). This is useful information when giving matricial presentations of algebraic structures, that provide the fundamental steps in the results below. As an application of our main theorem we use matrix problem techniques, as presented by Bautista, Salmerón and Zuazua [2] to produce tree presentations of diverse matrix problems, such as representations of posets and some minimal projective presentations over finite dimensional k-algebras (Section 4). We also give an alternative proof for Ringel's tree presentations [1] of the postprojective Kronecker modules (Section 5). These methods can be easily adapted to give tree presentations for nonexceptional modules (see [4,5] and Remark 4 below).

1. Preliminaries

Let $Q = (Q_0, Q_1, s, t)$ be a finite quiver (oriented graph). For an arbitrary field k consider the path k-algebra kQ. Its underlying vector space has as basis the paths in Q, with product determined by concatenation of paths. Assume there is a degree function $| - | : Q_1 \to \{0, 1\}$ in the set of arrows of Q. An arrow with degree zero (resp. one) is represented graphically as a solid (resp. dotted) arrow and $(Q, | - |)$ is called a biquiver. Then $kQ = \bigoplus_{i \geq 0} [kQ]_i$ is a graded algebra, where the degree of a path $a_m \cdots a_1$ is defined by $|a_m \cdots a_1| = \sum_{i=1}^{m} |a_i|$ and $[kQ]_i$ is the homogeneous subspace of degree i. Consider the subalgebra R of kQ generated by the trivial paths $e_1, \ldots, e_n$, which have degree zero and are in bijection with the vertices of Q. Then R is a trivial k-algebra, that is, a finite product of copies of the field k. Let W be the R–R-subbimodule of kQ generated by the arrows of Q (recall that the algebra kQ is freely generated by the pair (R, W) [2, 1.1]). The bimodule of arrows admits a graded decomposition $W = W_0 \oplus W_1$, where W_i is the bimodule generated by arrows of degree i, for $i \in \{0, 1\}$. The algebra kQ can be identified with the tensor algebra $T_R(W)$ with graded structure such that the elements of W_0 have degree 0 and those of W_1 have degree 1 (cf. [2, 4.1]). A differential δ on a graded algebra $T = \bigoplus_{i \geq 0} T_i$ is a linear transformation $\delta : T \to T$ such that $\delta(T_i) \subseteq T_{i+1}$ for all $i \geq 0$, and δ satisfies the Leibniz rule: $\delta(ab) = \delta(a)b + (-1)^{|a|} a\delta(b)$ for any pair of homogeneous elements a, b. A **differential quiver algebra** $\mathcal{A} = (kQ, \delta)$ consists of the path algebra kQ of a finite biquiver Q and a differential δ on kQ, such that