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An optimization problem concerning multiplicative functions

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ABSTRACT

In this paper we study the problem of maximizing a quadratic form $\langle Ax, x\rangle$ subject to $\|x\|_q = 1$, where A has matrix entries $f(\frac{[i,j]}{(i,j)})$ with $i, j|k$ and $q \geq 1$. We investigate when the optimum is achieved at a 'multiplicative' point; i.e. where $x_1 x_{mn} = x_m x_n$. This turns out to depend on both f and q, with a marked difference appearing as q varies between 1 and 2. We prove some partial results and conjecture that for f multiplicative such that $0 < f(p) < 1$, the solution is at a multiplicative point for all $q \geq 1$.

1. Introduction

In optimization problems involving multiplicative structure, there is a tendency for multiplicative functions to play a crucial role. This can appear in various ways; the optimum may itself be multiplicative, or the point where the optimum occurs may be multiplicative.

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For instance in [3], Codecá and Nair considered (amongst others) the problem of minimizing a quadratic form $\langle Bx, x\rangle$ subject to $\|x\|_2 = 1$ where B is the $d(k) \times d(k)$ matrix with entries $\frac{h((i,j))}{ij}$ where $i, j|k$, (i, j) is the gcd of i and j, and k is squarefree. They proved that any real multiplicative function f with $0 < f(p) < 1$ (for primes $p|k$) can be realised as such as minimum. Further, they explicitly determined this minimum when h is multiplicative and of the form $h = 1 * g$, with $g \geq 0$.

Another example comes from [7], where Perelli and Zannier considered the problem of minimizing $\langle Ax, x\rangle$ subject to $\|x\|_2 = 1$ where A is the $d(k) \times d(k)$ matrix (again with k squarefree) with entries $f(\frac{[i,j]}{(i,j)})$ (here $i, j|k$ and $[i, j]$ is the lcm of i and j) in the special case that $f(n) = \frac{1}{4} + \frac{1}{12n}$. They show that the minimum is $\frac{\varphi(k)}{12k}$ and that this is achieved at the point $x_d = \frac{\mu(d)}{\sqrt{d(k)}}$.

In [6], it was noted that the operation $c \circ d = \frac{[c,d]}{(c,d)}$ is a group operation on $D(k) = \{d : d|k\}$ if k is squarefree and, as an application of this algebraic structure, the problem of maximizing $\langle A_f x, x\rangle$ was considered, where $A_f = (f(c \circ d))_{c,d|k}$ but now subject to $\|x\|_q = 1$ with $q \geq 2$. It was found that for any $f : D(k) \to (0, \infty)$, the optimum is

$$d(k)^{1-\frac{2}{q}} \sum_{d|k} f(d),$$

and that it occurs at x_d constant. Notice that in both of the above examples, $\frac{x_d}{x_1}$ is multiplicative at the optimum, even if f is not. In the latter, the optimum itself is also multiplicative precisely when f is. Also in [3], the optimum can be shown to occur at multiplicative $\frac{x_d}{x_1}$.

In this paper we consider the above optimization problem for the range $1 < q < 2$, which turns out to be highly non-trivial. This has its origin in a problem concerning gcd sums. Briefly, one wishes to maximize the sum

$$F_\alpha(S) = \sum_{m,n\in S} \frac{1}{(m \circ n)^\alpha}$$

over all sets S of size N (see [5] for the case $\alpha = 1$ and [4] and [1] for other values of $\alpha > 0$). For $\alpha \geq \frac{1}{2}$, good bounds for this maximum have been established (sharp for $\alpha = 1$ [5] and close to best possible for $\frac{1}{2} \leq \alpha < 1$ see [1,2]), but for $0 < \alpha < \frac{1}{2}$ little is as yet known, except for rather crude upper and lower bounds. Thus it is known that in this range

$$N^{2-2\alpha} \ll \max_{|S|=N} F_\alpha(S) \ll N^{2-2\alpha} \exp\left\{ c\alpha \sqrt{\frac{\log N \log\log\log N}{\log\log N}} \right\}$$

for some absolute constant c (see [2]), but the true order is far from settled. In work in progress, a new lower bound $N^{2-2\alpha}(\log\log N)^{2\alpha}$ can be established which may also turn out to be the correct order of magnitude. This hinges (in part) on maximizing $\langle A_f x, x\rangle$