Linear Algebra and its Applications 485 (2015) 345–358

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Invariant factors of products over elementary divisor domains

Cristina Caldeira [1], João Filipe Queiró *,[2]

CMUC, Department of Mathematics, University of Coimbra, P-3001 454 Coimbra, Portugal

ARTICLE INFO

Article history:
Received 20 May 2015
Accepted 28 July 2015
Available online 11 August 2015
Submitted by R. Brualdi

MSC:
15B33
15B36

Keywords:
Invariant factors
Principal ideal domains
Elementary divisor domains

ABSTRACT

For matrices A and B, what can we say about the invariant factors of AB in terms of those of A and B? For matrices over principal ideal domains, the complete answer is known. In the present paper we consider the same problem for matrices over the larger class of elementary divisor domains.

1. Introduction

In this paper we are interested in describing the invariant factors of the product of two matrices over the most general class of integral domains for which the question makes

* Corresponding author.
E-mail addresses: caldeira@mat.uc.pt (C. Caldeira), jfqueiro@mat.uc.pt (J.F. Queiró).

[1] This research was done within the activities of the Centre for Mathematics of the University of Coimbra.
[2] This work was partially supported by the Centre for Mathematics of the University of Coimbra – UID/MAT/00324/2013, funded by the Portuguese Government through FCT/MEC and co-funded by the European Regional Development Fund through the Partnership Agreement PT2020.

http://dx.doi.org/10.1016/j.laa.2015.07.038
0024-3795/

sense. The problem has been completely solved for matrices over principal ideal domains (PIDs) and we begin in that setting. There is no loss of generality in restricting our study to square nonsingular matrices [14].

Let R be a PID and A an $n \times n$ nonsingular matrix over R. It is well known that A is equivalent to its *Smith normal form*, that is, there exist U and V unimodular (*i.e.* invertible over R) such that

$$UAV = \begin{bmatrix} a_n & 0 & \cdots & 0 \\ 0 & a_{n-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_1 \end{bmatrix},$$

where $a_n \mid a_{n-1} \mid \cdots \mid a_1$ are the *invariant factors* of A.

The invariant factors are uniquely determined by A, as follows from the characterization

$$a_{n-k+1} = \frac{d_k(A)}{d_{k-1}(A)}\ ,\ \ k = 1, \ldots, n,$$

where, for each k, $d_k(A)$, the so-called kth *determinantal divisor* of A, is the gcd of all $k \times k$ minors of A, $d_0 \equiv 1$. (This definition can of course be presented also for non-square matrices.) By the Cauchy–Binet theorem for determinants, the d_k are invariant under equivalence. That $d_{k-1}(A)$ divides $d_k(A)$ follows from Laplace's theorem.

The problem we are interested in is the following: What are the possible invariant factors $c_n \mid \cdots \mid c_1$ of a product AB, if A and B are $n \times n$ nonsingular matrices over R with invariant factors $a_n \mid \cdots \mid a_1$ and $b_n \mid \cdots \mid b_1$, respectively?

For matrices over a PID, this problem has been solved with a variety of approaches, starting with its p-module version in [10], where p is a prime in R. Indeed, all approaches start by localizing the problem at an arbitrary prime p, working in that context, and then recovering the global solution.

To describe the solution in [10] we need some notation. For each fixed prime $p \in R$, we restrict our attention to matrices over the local ring R_p, that is, we just work with powers of p: $a_i \to p^{\alpha_i}$, $b_i \to p^{\beta_i}$, $c_i \to p^{\gamma_i}$, where $\alpha_1 \geq \cdots \geq \alpha_n$, $\beta_1 \geq \cdots \geq \beta_n$, $\gamma_1 \geq \cdots \geq \gamma_n$ are nonnegative integers.

Denote by $IF(\alpha, \beta)$ the set of possible γ in the invariant factor product problem. Introduce the notation $\Lambda_n = \{\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n : \alpha_1 \geq \cdots \geq \alpha_n \geq 0\}$. What was proved in [10] was that $IF(\alpha, \beta) = LR(\alpha, \beta)$, where the latter is the set of $\gamma \in \Lambda_n$ which can be obtained from α and β using the combinatorial Littlewood–Richardson rule (for the description of the rule see *e.g.* [6]). Thus the invariant factor product problem, in its local "primary" version, has a complete and interesting solution, although not a clearly explicit one, via the Littlewood–Richardson rule. In particular, this solution is *not* given as a family of divisibility relations.