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Nonsymmetric normal entry patterns with the maximum number of distinct indeterminates

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ABSTRACT

We prove that a nonsymmetric normal entry pattern of order n ($n \geq 3$) has at most $n(n-3)/2+3$ distinct indeterminates and up to permutation similarity this number is attained by a unique pattern which is explicitly described. An open problem is posed.

1. Introduction

Symmetric matrices, Toeplitz matrices, Hankel matrices and circulant matrices all require repetitions of some entries in prescribed positions. These special matrices suggest that we define a new concept for investigation of the general situation. Given a set S, we

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denote by $M_n(S)$ the set of $n \times n$ matrices whose entries are from S. If $S = \{x_1, \dots, x_k\}$ is a finite set, we write $M_n\{x_1, \dots, x_k\}$ for $M_n(S)$.

Definition 1. Let $x_1, x_2, \dots, x_k$ be distinct indeterminates. We call a matrix in $M_n\{x_1, x_2, \dots, x_k\}$ an *entry pattern.*

Thus an entry pattern is a matrix whose entries are indeterminates some of which may be equal. For example, among

$$A = \begin{bmatrix} x & y \\ y & z \end{bmatrix}, \quad B = \begin{bmatrix} 2x & x+y \\ -z & w \end{bmatrix}, \quad C = \begin{bmatrix} 3 & x \\ y & z \end{bmatrix}$$

A is an entry pattern while B and C are not. Rectangular entry patterns are defined similarly.

The spirit of entry patterns is that sometimes we can deduce properties of certain special matrices by just looking at the patterns of their entries without knowing the actual entries. This is possible. For example, every real symmetric matrix has all real eigenvalues and every complex circulant matrix is normal [4, p. 5]. Entry patterns will serve the study of matrices over fields. To avoid unnecessary technical complications we consider only real matrices.

The *pattern class* of an entry pattern A, denoted $A(\mathbb{R})$, is the set of the real matrices obtained by specifying the values of the indeterminates of A. Thus

$$\begin{bmatrix} 2 & 3 & 5 \\ 5 & 2 & 3 \\ 3 & 5 & 2 \end{bmatrix} \in A(\mathbb{R}) \ \text{ with } \ A = \begin{bmatrix} x & y & z \\ z & x & y \\ y & z & x \end{bmatrix}.$$

Conversely,

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 2 & 4 & 3 \end{bmatrix} \ \text{and} \ \begin{bmatrix} 5 & 5 & 5 \\ 6 & 7 & 8 \\ 6 & 8 & 7 \end{bmatrix} \ \text{have the same entry pattern} \ \begin{bmatrix} x & x & x \\ y & z & w \\ y & w & z \end{bmatrix}.$$

We denote by A^T the transpose of a matrix A. Recall that a real matrix A is said to be *normal* if $AA^T = A^T A$. Including symmetric matrices and orthogonal matrices as subclasses, normal matrices have nice properties and they are an important topic in matrix analysis. See [1, Chapters VI and VII] and [2, Chapter 8].

Definition 2. A square entry pattern A is said to be *normal* if every matrix in $A(\mathbb{R})$ is normal.

Symmetric entry patterns are obviously normal. There are many nonsymmetric normal entry patterns. We will determine the maximum number of distinct indeterminates