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# A projection and an effect in a synaptic algebra

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We study a pair $p, e$ consisting of a projection $p$ (an idempotent) and an effect $e$ (an element between 0 and 1) in a synaptic algebra (a generalization of the self-adjoint part of a von Neumann algebra). We show that some of Halmos's theory of two projections (or two subspaces), including a version of his CS-decomposition theorem, applies in this setting, and we introduce and study two candidates for a commutator projection for $p$ and $e$.



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## 1. Introduction

In [16], P. Halmos studied two projection operators $P$ and $Q$ on a Hilbert space and proved a basic theorem, now called the *CS-decomposition theorem*, that expresses $Q$ in terms of $P$ and positive contraction operators $C$ and $S$, called the *cosine* and the *sine* operators, respectively, for $Q$ with respect to $P$. For a lucid and extended exposition of Halmos's theory of two projections, see [2]. In [12], we proved a generalization of the CS-decomposition theorem in the setting of a so-called synaptic algebra [12, Theorem 5.6].

In what follows, $A$ is a synaptic algebra with enveloping algebra $R \supseteq A$ [3,7,9–11, 21], $P$ is the orthomodular lattice [1,18] of projections in $A$, and $E$ is the convex effect algebra [4,14] of all effects in $A$. To help fix ideas, we note that the self-adjoint part of a von Neumann algebra, and more generally of an AW$^*$-algebra, forms a synaptic algebra. Numerous additional examples are given in the literature cited above.

In this article we generalize the CS-decomposition theorem for two projections $p, q \in P \subseteq A$ to the case of a projection $p \in P$ and an effect $e \in E$ (Theorem 3.9 below), and we investigate two candidates for the commutator projection for the pair $p$ and $e$ (Section 5 below).

In our generalization of the CS-decomposition theorem, which we call the *CBS-decomposition theorem*, the cosine and sine effects $c$ and $s$ introduced in [12, Definition 4.2] are generalized (Definition 3.1 below) and supplemented by a third effect $b$ (Definition 3.6 below).

Part of our motivation for the work in this article derives from our interest in *the infimum problem* as applied to the synaptic algebra $A$, i.e., the problem of determining just when two effects $e, f \in E$ have an infimum $e \wedge f$ in $E$, and if possible, finding a perspicuous formula for $e \wedge f$ when it does exist. That this problem is non-trivial is indicated by a remark of P. Lahti and M. Mączynski in [19, p. 1674] that the partial order structure of $E$ is "rather wild." The development in [15] and [20] suggests that it might be possible to make progress on the infimum problem for $A$ if the problem can be solved for the pair $p, e$ with $p \in P$ and $e \in E$. We hope that our results in this article will cast some light on the latter problem. In Section 6 below, we illustrate the utility of the CBS-decomposition theorem by applying it to generalize a result of T. Moreland and S. Gudder concerning the infimum problem [20] to the setting of a synaptic algebra.

## 2. Some basic definitions, notation, and facts

In this section we briefly outline some notions that we shall need below. For the definition of a synaptic algebra and more details, see the literature cited above, especially [3] and [10]. In what follows, the notation := means 'equals by definition,' the ordered field of real numbers and its subfield of rational numbers are denoted by $\mathbb{R}$ and $\mathbb{Q}$, and 'iff' abbreviates 'if and only if.'