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A characterization of oriented hypergraphic balance via signed weak walks



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ABSTRACT

An oriented hypergraph is a hypergraph where each vertex-edge incidence is given a label of $+1$ or -1 , and each adjacency is signed the negative of the product of the incidences. An oriented hypergraph is *balanced* if the product of the adjacencies in each circle is positive.

We provide a combinatorial interpretation for entries of k th power of the oriented hypergraphic Laplacian via the number of signed weak walks of length k . Using closed weak walks we prove a new characterization of balance for oriented hypergraphs and matrices that generalizes Harary's Theorem for signed graphs.

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1. Introduction

An *oriented hypergraph* is a signed incidence structure where each vertex-edge incidence is given a label of $+1$ or -1 , and each adjacency is signed the negative of their incidence product. A *signed graph* is an oriented hypergraph where each edge is con-

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tained in at most two incidences, while a graph can be considered as a signed graph where all adjacencies are positive and each edge is contained in exactly two incidences. An oriented hypergraph is said to be *balanced* if the product of the adjacencies in each circle is positive. In Harary's seminal paper on signed graphs in 1953 he provided the following equivalent condition for balance in signed graphs to model social interactions using signed paths:

Theorem 1.1. (See Harary [6].) *A signed graph is balanced if, and only if, for each pair of vertices v and w all vw -paths have the same sign.*

Harary's Theorem is not true for hypergraphs with edges of size 3 or greater. We relax the path condition and provide a non-trivial oriented hypergraphic generalization of Harary's Theorem.

The concept of a balance, as well as identifying equivalent conditions, is critical to the structure of many combinatorial optimization and programming, for a proper introduction see [4,10]. The (non-oriented) balanced hypergraph was introduced by Berge in 1970 [1] as one of a number of different generalizations of bipartite graphs; this was further generalized to balanced $\{0, \pm 1\}$ -matrices by Truemper in 1982 [13], and to incidence oriented hypergraphs by Shi in 1992 [12] and Rusnak in 2013 [9]. The benefit of working with the current oriented hypergraphic model is that it unifies Harary's work on signed graphs with the current theory of balanced matrices, and provides a translation between many graphic, hypergraphic, and balanced matrix theorems.

It was shown by Fulkerson et al. [5] that the condition of balance in $\{0, 1\}$ -matrices is equivalent to integrality of set covering, set packing, and set partitioning polytopes as well as total dual integrality of linear systems, these results were extended to $\{0, \pm 1\}$ -matrices by Conforti and Cornuéjols in [2]. Berge also provided the following characterization of balance, which was also generalized to balanced $\{0, \pm 1\}$ -matrices by Conforti and Cornuéjols in [2]:

Theorem 1.2. (See Berge [1].) *A $\{0, 1\}$ -matrix $\mathbf{M}$ is balanced if, and only if, every sub-matrix of $\mathbf{M}$ is bicolourable.*

These generalizations can be trivially incorporated into oriented hypergraphs by regarding the given matrix as the incidence matrix of an oriented hypergraph. However, a generalization of Harary's Theorem would provide a characterization of balance via new hypergraphic structures and avoid examining matrices. A structural characterization of obstructions to balance was given by Truemper in [14], while Conforti, Cornuéjols, and Rao's famous work [3] on recognizing balanced $\{0, 1\}$ -matrices in polynomial time won the 2000 Fulkerson Prize. More recently balanced matrices were generalized to oriented hypergraphs to examine integrated circuits and various applications to VLSI via minimization [11,12]. Additionally, the concept of balance is central to the characterization of the matroid structure of signed graphs [15–17] as well as oriented hypergraphs as introduced in [9].

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