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Monotonically positive matrices

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ABSTRACT

A matrix A is monotonically positive (MP) if there exists a matrix U such that $A = UU^T$ and each column of U is monotonically nonincreasing or nondecreasing. Following the approach in J. Nie (2014) [15], we propose a semidefinite algorithm for checking whether or not a matrix is MP. If it is not MP, a certificate for it can be obtained; if it is MP, an MP-decomposition can be obtained. Some computational experiments are presented to show how to do this.

1. Introduction

Suppose that $(X_1, \dots, X_n)$ are n jointly distributed random variables. Order statistics studies properties of the $X_i's$ that appear in the nondecreasing order

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$X_1 \le X_2 \le \cdots \le X_n$. We refer to [1,5] for the theory of order statistics. Suppose x is a random vector in $\mathbb{R}^n$ with the expectation $\mathbb{E}x = b$ and the covariance matrix C, where

$$C_{ij} = \mathbb{E}[(x_i - b_i)(x_j - b_j)], \quad i, j = 1, \ldots, n. \tag{1}$$

Let A be the matrix with

$$A_{ij} = \mathbb{E}(x_i x_j) = C_{ij} + b_i b_j, \quad i, j = 1, \ldots, n.$$

In the area of order statistics, a basic and natural problem is to study the probability function of x that is supported in $\mathbb{MR}^n$, where $\mathbb{MR}^n$ is the cone of monotonically nondecreasing vectors:

$$\mathbb{MR}^n = \{x \in \mathbb{R}^n \mid x_1 \le \cdots \le x_n\}.$$

The above problem is equivalent to whether there exists a finite atomic Borel measure μ supported in $\mathbb{MR}^n$ such that

$$A = \int_{\mathbb{MR}^n} x x^T d\mu = \sum_{i=1}^{m} u_i u_i^T,$$

where $u_i \in \mathbb{MR}^n (i = 1, \ldots, m)$ are support points.

Considering the above application, we propose the new notion of monotonically positive matrices. A real $n \times n$ symmetric matrix A is monotonically positive (MP) if there exist vectors $u_1, \cdots, u_m \in \mathbb{MR}^n$ or $-u_1, \cdots, -u_m \in \mathbb{MR}^n$ such that

$$A = u_1 u_1^T + \cdots + u_m u_m^T, \tag{2}$$

where m is called the length of the decomposition (2). The smallest m in the above is called the MP-rank of A. If A is monotonically positive, we call (2) an MP-decomposition of A. Clearly, a matrix A is monotonically positive if and only if $A = UU^T$ such that each column of U is monotonically nondecreasing or nonincreasing. Obviously, an MP-matrix is always positive semidefinite.

For example, consider the matrix A given as

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{pmatrix}. \tag{3}$$

Obviously,

$$A = uu^T = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}^T.$$