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Proof of a conjectured lower bound on the chromatic number of a graph

Tsuyoshi Ando [a], Minghua Lin [b,c,*]

[a] Hokkaido University (Emeritus), Sapporo, Japan
[b] Department of Mathematics, Shanghai University, Shanghai 200444, China
[c] Department of Mathematics and Statistics, University of Victoria, Victoria, BC, V8W 3R4, Canada

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ABSTRACT

We confirm a conjecture in Wocjan and Elphick (2013) [4] about a lower bound of the chromatic number of a graph.

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1. Introduction

In this note, we let G be a simple graph with $n \geq 1$ vertices and let A_G be its adjacency matrix. The chromatic number $\chi(G)$ of the graph G is the smallest number of colors needed to color the vertices of G so that no two adjacent vertices share the same color, i.e., the smallest value of r possible to obtain a r-coloring.

* Corresponding author at: Department of Mathematics, Shanghai University, Shanghai 200444, China.
E-mail addresses: ando@es.hokudai.ac.jp (T. Ando), mlin87@ymail.com (M. Lin).

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Obviously, if $r = \chi(G)$, then the vertices of G can be numbered in such a way that A_G is partitioned into $r \times r$ blocks $A_G = [A_{ij}]_{i,j=1}^r$ with $A_{11}, \dots, A_{rr}$ being zero matrices. Though the diagonal blocks of A_G may be of different size, they are required to be square matrices.

The adjacency matrix A_G is real symmetric, so the eigenvalues of A_G are real numbers, which we denote by $\mu_1, \dots, \mu_n$, sorted in non-increasing order. The Hoffman lower bound on the chromatic number is a classical result in spectral graph theory

$$\chi(G) \geq 1 + \frac{\mu_1}{-\mu_n}.$$

Over the years, there has been many studies on finding reasonable lower bounds of $\chi(G)$; see, e.g., [2–4] and references therein.

Let the inertia of A_G be (π, ν, δ), where π, ν and δ are the numbers (counting multiplicities) of positive, negative and zero eigenvalues of A_G respectively. Let

$$s^+ = \mu_1^2 + \cdots + \mu_\pi^2,$$

and

$$s^- = \mu_{n-\nu+1}^2 + \cdots + \mu_n^2.$$

In [4], the following conjecture was made

Conjecture 1.1. *Let s^+, s^- be defined as above. Then*

$$\chi(G) \geq 1 + \frac{s^+}{s^-}.$$

Although the conjecture has been confirmed for various graph families [4], a complete solution seems not known. The purpose of this note is to provide a complete solution. The main techniques used in this note are from matrix analysis [1], so it is an evidence of fruitful interplay between combinatorics and linear algebra.

2. Main result

In the sequel, the norm we consider is the Frobenius norm. That is, for a complex matrix X, $\|X\| = \sqrt{\mathrm{tr} X^\dagger X}$, where $X^\dagger$ means the conjugate transpose of X and tr means the trace. It is clear that if X is Hermitian, then $\|X\|^2$ is equal to the sum of the squares of all the eigenvalues of X. For a block matrix, the diagonal blocks are always assumed to be square.

We need the following simple lemma.

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