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Convexity of tropical polytopes

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ABSTRACT

We study the relationship between min-plus, max-plus and Euclidean convexity for subsets of $\mathbb{R}^n$. We introduce a construction which associates to any max-plus convex set with compact projectivisation a canonical matrix called its *dominator*. The dominator is a Kleene star whose max-plus column space is the min-plus convex hull of the original set. We apply this to show that a set which is any two of (i) a max-plus polytope, (ii) a min-plus polytope and (iii) a Euclidean polytope must also be the third. In particular, these results answer a question of Sergeev, Schneider and Butkovič [16] and show that row spaces of tropical Kleene star matrices are exactly the "polytropes" studied by Joswig and Kulas [13].

The notion of *tropical convexity* (also known as *max-plus* or *min-plus* convexity) has long played an important role in max-plus algebra and its numerous application areas (see for example [3,4]). More recently, applications have emerged in areas of pure mathematics as diverse as algebraic geometry [7] and semigroup theory [9,10].

Recall that a subset of $\mathbb{R}^n$ is called *max-plus convex* if it is closed under the operations of componentwise maximum ("max-plus sum") and of adding a fixed value to each

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component ("tropical scaling"). A *max-plus polytope* is a non-empty max-plus convex set which is generated under these operations by finitely many of its elements; max-plus polytopes are exactly the row spaces (or column spaces) of matrices over the max-plus semiring. There are obvious dual notions of min-plus convexity and min-plus polytopes. Min-plus and max-plus polytopes are sometimes called *tropical polytopes*.[1] We note that these objects may be thought of either as polytopes in projective space or as finitely generated convex cones in affine space. We have preferred the shorter term (polytope) for consistency with [7,13], but note that these objects have also been termed "finitely generated tropical cones" elsewhere in the literature.

The *projectivisation* of a max-plus polytope is the set of orbits of its points under the action of tropical scaling. Since any point can be scaled to put 0 in the first coordinate (say), restricting the polytope to points with first coordinate 0 gives a cross-section of the scaling orbits, and hence a natural identification of the projectivisation with a subset of $\mathbb{R}^{n-1}$. A subset which arises from a max-plus polytope in this way we term a *projective max-plus polytope*. In general, a projective max-plus polytope is a compact Euclidean polyhedral complex in $\mathbb{R}^{n-1}$; it may or may not be a convex set in the ordinary Euclidean metric. Joswig and Kulas [13] studied the class of projective max-plus polytopes[2] which *are* Euclidean convex, and hence also Euclidean polytopes. These sets, which they term *polytropes*, turn out to have numerous interesting properties and applications (see for example [6,17,18]).

At around the same time, Sergeev, Schneider and Butkovič [16] studied the tropical polytopes arising as row spaces of *Kleene stars*. Kleene stars are a class of particularly well-behaved idempotent max-plus matrices which play a vital role in almost all aspects of max-plus algebra (see Section 1 below for a definition and [3] for a comprehensive introduction). They prove, among many other interesting things, that the column space of a max-plus[3] Kleene star is always Euclidean convex [16, Propositions 2.6 and 3.1]; since such column spaces are max-plus polytopes by definition, this means they are all examples of Joswig–Kulas "polytropes". The question of whether the converse holds was posed (with a slightly differently phrasing) in [16, p. 2400]: is a Euclidean-convex max-plus polytope necessarily the column space of a Kleene star? One aim of the present paper is to answer this question in the affirmative, establishing that "polytropes" are exactly column (and row) spaces of Kleene stars. This unifies the research in [13] and [16],

[1] Typically one fixes upon either the "min convention" or the "max convention" and uses terms such as "tropically convex" and "tropical polytope" to refer to min-plus or max-plus as appropriate. A key feature of this paper is that we study the relationship *between* min-plus and max-plus convexity, so for the avoidance of ambiguity we will tend not to use the word "tropical".

[2] Formally speaking they studied projective *min-plus* polytopes, but the difference is immaterial because *negation* of vectors forms a trivial duality between min-plus and max-plus convex sets while preserving Euclidean convexity.

[3] In fact [16] is written in the language of max-*times* algebra and geometry; this is exactly equivalent to max-plus via the logarithm map, save for the fact that they work with a 0 element, which corresponds to a $-\infty$ element in the max-plus case. For simplicity we work here without $-\infty$, but the results of [16] all specialise to apply in this case. The concept corresponding to Euclidean convexity in the max-times setting is *log-convexity*.