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Tensor–tensor products with invertible linear transforms

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Research in tensor representation and analysis has been rising in popularity in direct response to a) the increased ability of data collection systems to store huge volumes of multidimensional data and b) the recognition of potential modeling accuracy that can be provided by leaving the data and/or the operator in its natural, multidimensional form. In recent work [1], the authors introduced the notion of the t-product, a generalization of matrix multiplication for tensors of order three, which can be extended to multiply tensors of arbitrary order [2]. The multiplication is based on a convolution-like operation, which can be implemented efficiently using the Fast Fourier Transform (FFT). The corresponding linear algebraic framework from the original work was further developed in [3], and it allows one to elegantly generalize all classical algorithms from numerical linear algebra. In this paper, we extend this development so that tensor–tensor products can be defined in a so-called transform domain for *any* invertible linear transform. In order to properly motivate this transform-based approach, we begin by defining a new tensor–tensor product alternative to the t-product. We then show that it can be implemented efficiently using DCTs, and that subsequent definitions and factorizations can be formulated by appealing to the transform domain. Using this new product as our guide, we then generalize the transform-based approach to any

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invertible linear transform. We introduce the algebraic structures induced by each new multiplication in the family, which is that of C*-algebras and modules. Finally, in the spirit of [4], we give a matrix–algebra based interpretation of the new family of tensor–tensor products, and from an applied perspective, we briefly discuss how to choose a transform. We demonstrate the convenience of our new framework within the context of an image deblurring problem and we show the potential for using one of these new tensor–tensor products and resulting tensor-SVD for hyperspectral image compression.

1. Introduction

Recently, the t-product has been introduced as a useful generalization of matrix multiplication for tensors of order three (i.e. three-way arrays) [1] and higher [2]. The t-product results in sensible definitions of transpose and orthogonality, and it allows for QR- and SVD-like decompositions [1], where the latter can be leveraged to derive an Eckart–Young like theorem for third-order tensors. Many matrix algorithms carry over simply by replacement of traditional matrix multiplication with the t-product [4]. The t-product has been applied to image deblurring [1], face recognition [5], completion of seismic data [6], and tensor compression [7].

As described in detail in [1], the t-product of a pair of third-order tensors is defined by unfolding tensors into block circulant matrices, multiplying the matrices, and folding the result back up into a third-order tensor. Furthermore, it is shown that the t-product can be computed efficiently by performing a discrete Fourier transform along the tube fibers of each tensor (i.e. into the page), performing pair-wise matrix products for all frontal faces of the tensors in the "transform domain" and then applying an inverse DFT along the tube fibers of the result. In [3,4,8], the authors exploit the fact that the t-product between $1 \times 1 \times n$ tensors is in fact equivalent to the discrete convolution of two vectors. Thus, the t-product between a pair of third-order tensors amounts to the traditional matrix multiplication algorithm between two matrices whose entries are $1 \times 1 \times n$ tensors, where the usual scalar product is replaced by discrete convolution. In [9], this viewpoint was extended using convolutions indexed by abstract finite groups.

The t-product has a disadvantage in that for real tensors, implementation of the t-product and factorizations using the t-product require intermediate complex arithmetic which, even taking advantage of complex symmetry in the Fourier domain, is more expensive than real arithmetic. It has been shown that convolution multiplication properties extend to different types of convolution [10], and there are corresponding structured matrix diagonalization results involving real-valued fast transforms [11,12]. Some of these transforms, such as the discrete cosine transform, are known to be highly useful in image processing [13]. Hence, following along a similar line of development as for the t-product,