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A note on connected bipartite graphs of fixed order and size with maximal index

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ABSTRACT

In this paper the unique graph with maximal index (i.e. the largest eigenvalue of the adjacency matrix) is identified among all connected bipartite graphs of order n and size $n+k$, under the assumption that $k \geq 0$ and $n \geq k+5$.

1. Introduction

Let $G = (V, E)$ be a simple graph with vertex set V and edge set E. Its order is $|V|$, denoted by n, and its size is $|E|$, denoted by m. Let $A = A(G)$ be the $(0,1)$-adjacency matrix of G. Since A is symmetric, its eigenvalues are real, and also called the eigenvalues

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of G. The largest eigenvalue of G is denoted by $\rho(G)$, and also called the *spectral radius*, or *index* for short. For the least eigenvalue of G we write $\lambda(G)$. If clear from the context, graph names are usually omitted. For all other terminology and notation the reader is referred to [9].

In [3], Bell et al. studied connected graphs whose least eigenvalue is minimal among graphs of prescribed order and size. Their main structural result reads:

Theorem 1. *Let G be a connected graph whose least eigenvalue is minimal among the connected graphs of order n and size m $(n-1 \leq m < \binom{n}{2})$. Then G is*

(i) a bipartite graph, or
(ii) a join (or complete product) of two nested split graphs (not both totally disconnected).

Recall, a graph is called a *nested split graph* (or *NSG* for short) if its vertices can be ordered so that $jq \in E(G)$ implies $ip \in E(G)$ whenever $i \leq j$ and $p \leq q$. Nested split graphs are in fact threshold graphs, so $\{2K_2, P_4, C_4\}$-free graphs.

In [4] and [11], further steps have been made in investigating graphs G for which the least eigenvalue $\lambda(G)$ is minimal among connected graphs of prescribed order and size. Namely, the structure of connected bipartite graphs of prescribed order and size with maximal index is studied, and thereby the structure of those with minimal least eigenvalue. The relevance of these investigations stems from Theorem 1(i), and well-known fact that $\lambda(G) = -\rho(G)$ for any bipartite graph G (see, e.g. [9], p. 56). Before we state the main result from [4] and [11] we first introduce a further class of bipartite graphs, namely *double nested graphs* (also called *bipartite chain graphs*, see for example [5]). As is well known from the literature, these graphs are $\{2K_2, C_3, C_5\}$-free graphs (see, for example, [2]).

Let G be a connected bipartite graph with colour classes U and V. We say that G is *double nested graph* (or *DNG* for short) if there exist partitions

$$U = U_1 \,\dot\cup\, U_2 \,\dot\cup\, \ldots \,\dot\cup\, U_h \quad \text{and} \quad V = V_1 \,\dot\cup\, V_2 \,\dot\cup\, \ldots \,\dot\cup\, V_h,$$

such that the neighbourhood of each vertex in U_1 is $V_1 \,\dot\cup\, V_2 \,\dot\cup\, \ldots \,\dot\cup\, V_h$, the neighbourhood of each vertex in U_2 is $V_1 \,\dot\cup\, V_2 \,\dot\cup\, \ldots \,\dot\cup\, V_{h-1}$, and so on. If $|U_i| = m_i$ and $|V_i| = n_i$ $(i = 1, 2, \ldots, h)$ then we write

$$G = D(m_1, m_2, \ldots, m_h; n_1, n_2, \ldots, n_h). \tag{1}$$

Such a graph is depicted in Fig. 1. Any fat circle corresponds to a co-clique of an appropriate size; any line between two fat circles means that each vertex in one fat circle is adjacent to all vertices in the other fat circle.

For the double nested graphs the next two relations hold:

$$n(G) = m_1 + \cdots + m_h + n_1 + \cdots + n_h\,, \tag{2}$$

$$m(G) = m_1(n_1 + \cdots + n_h) + m_2(n_1 + \cdots + n_{h-1}) + \cdots + m_h n_1\,. \tag{3}$$