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Linear subspaces of matrices associated to a Ferrers diagram and with a prescribed lower bound for their rank

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Fix a field K, a subset $\mathcal{P} \subseteq \{1, \ldots, k\} \times \{1, \ldots, m\}$ and an integer $\delta \leq \min\{k, m\}$. Let $C(m, k, \mathcal{P}, K)$ be the vector space of all $k \times m$ matrices with entries $a_{i,j} = 0$ if $(i, j) \notin \mathcal{P}$. Let $\alpha(\delta, K)$ be the maximal dimension of a linear subspace $V \subseteq C(m, k, \mathcal{P}, K)$ such that all $A \in V \setminus \{0\}$ have rank $\geq \delta$. We show that known lower bounds for $\alpha(\delta, K)$, for K one (resp. several, resp. almost all) finite field give the same lower bounds for some (resp. many, resp. all) number fields.

1. Introduction

For any positive integer m set $[m] := \{1, \ldots, m\}$. Fix positive integers k, m and a field K. A *configuration* or a *profile* of size $k \times m$ is a subset $\mathcal{P}$ of $[k] \times [m]$.

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Let $C(m,k,\mathcal{P},K)$ be the set of all $k \times m$ matrices $A = (a_{i,j})$ over K with $a_{i,j} = 0$ if $(i,j) \notin \mathcal{P}$. The set $C(m,k,\mathcal{P},K)$ is a K-vector space. For each integer δ with $0 \le \delta \le \min\{k,m\}$ let $C(m,k,\mathcal{P},K,\le \delta)$ (resp. $C(m,k,\mathcal{P},K,= \delta)$, resp. $C(m,k,\mathcal{P},K,\ge \delta)$) denote the set of all $A \in C(m,k,\mathcal{P},K)$ with rank $\le \delta$ (resp. $= \delta$, resp. $\ge \delta$). These sets are classically studied [2,4,5,9,13,14], often when K is algebraically closed (but [16] contains a lower bound $(m-\delta+1)^2$ if $m = k$ over any infinite field). In particular it is interesting to know the maximal linear subspaces of $C(m,k,\mathcal{P},K,*)$, $*$ either $\le$ or $=$ or $\ge$, and the maximal dimension of a linear subspace of $C(m,k,\mathcal{P},K,*)$, $*$ either $\le$ or $=$ or $\ge$.

We explain how these sets appeared in the applied mathematics and computer science literature (network coding [12]) when K is a finite field [6–8,10,16,17,19,20]. For any two $k \times m$ matrices A, B over a field K set $d_R(A,B) := \mathrm{rank}(A-B)$. The function d_R induces a metric on the set $M_{k\times m}(K)$ of all $k \times m$ matrices over K (it is called the rank metric). A $[k \times m, \rho, \delta]$ rank-metric code $\mathcal{C}$ over $\mathbb{F}_q$ is a linear space $\mathcal{C} \subseteq M_{k\times m}(\mathbb{F}_q)$ such that $\dim_{\mathbb{F}_q}(\mathcal{C}) = \rho$ and $d_R(A,B) \ge \delta$ for all $A, B \in \mathcal{C}$ (another convention would say that the same object $\mathcal{C}$ is a $[k \times m, \rho, \ge \delta]$ rank-metric code, because we do not require the existence of $A, B \in \mathcal{C}$ with $\mathrm{rank}(A-B) = \delta$). For rank-metric codes there is a Singleton bound $\rho \le \min\{m(k-\delta+1), k(m-\delta+1)\}$ and a notion of maximum distance MRD rank codes, i.e. the rank-metric codes whose minimum distance is equal to the Singleton bound (for the usual block codes they are the MDS codes) [3,7,16].

Since one expects improvements over finite fields (even machine computations for small fields, mainly even fields), we point out why any advance on finite fields may easily translated into similar results for number fields. The K-linear subspaces of $C(m,k,\mathcal{P},K,\ge \delta)$ are called $\underline{\delta}$-linear subspaces (these are the more interesting subspaces in network coding), while the K-linear subspaces of $C(m,k,\mathcal{P},K,\le \delta)$ are called $\bar{\delta}$-subspace (they also appear in the network code literature under the name of linear anticodes).

A configuration $\mathcal{F}$ of size $k \times m$ is called a *Ferrers diagram* if $\mathcal{F} \ne \emptyset$ and the following conditions are satisfied:

(1) if $(i,j) \in \mathcal{F}$ and $i > 1$, then $(i-1,j) \in \mathcal{F}$;
(2) if $(i,j) \in \mathcal{F}$ and $j < m$, then $(i,j+1) \in \mathcal{F}$.

Let $\mathrm{Maxdim}_{\underline{\delta}}(m,k,\mathcal{P},K)$ (resp. $\mathrm{Maxdim}_{\bar{\delta}}(m,k,\mathcal{P},K)$) denote the maximal dimension of a $\underline{\delta}$-subspace (resp. a $\bar{\delta}$-subspace). The integer $\mathrm{Maxdim}_{\bar{\delta}}(m,k,\mathcal{P},K)$ only depends on k, m, $\mathcal{P}$ and δ, but not on the field K [8, Theorem 46]. Therefore we may forget about them and only look at $\underline{\delta}$-linear subspaces. Our aim is to show that results over finite fields give results over number fields, i.e. over finite extensions of $\mathbb{Q}$. This idea is not new. Indeed, cyclic finite field extensions allow construction of $\underline{d}$-subspaces [9, Lemma 3.2] and these extensions are used in network coding (Gabidulin's rank-metric codes [7]). The case in which $\mathcal{P} = \{1,\dots,k\} \times \{1,\dots,m\}$ is known over $\mathbb{Q}$ and so if $\mathcal{P} = \{1,\dots,k\} \times \{1,\dots,m\}$ Theorem 1 was well-known.