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Minimum (maximum) rank of sign pattern tensors and sign nonsingular tensors

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ABSTRACT

In this paper, we define the sign pattern tenors, minimum (maximum) rank of sign pattern tenors, term rank of tensors and sign nonsingular tensors. The necessity and sufficiency for the minimum rank of sign pattern tenors to be 1 is given. We show that the maximum rank of a sign pattern tensor is not less than the term rank and the minimum rank of the sign pattern of a sign nonsingular tensor is not less than its dimension. We get some characterizations of tensors having sign left or sign right inverses.

1. Introduction

For a positive integer n, let $[n] = \{1, \dots, n\}$. Let $\mathbb{R}^{n_1 \times \cdots \times n_k}$ be the set of the k-order tensors over real field. A k-order tensor $\mathcal{A} = (a_{i_1 \cdots i_k}) \in \mathbb{R}^{n_1 \times \cdots \times n_k}$ is a multidimensional

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array with $n_1 \times n_2 \times \cdots \times n_k$ entries. When $k = 2$, $\mathcal{A}$ is an $n_1 \times n_2$ matrix. If $n_1 = \cdots = n_k = n$, then $\mathcal{A}$ is called a k-order n-dimension tensor. The k-order n-dimension tensor $\mathcal{I} = (\delta_{i_1 \cdots i_k})$ is called a unit tensor, where $\delta_{i_1 \cdots i_k} = 1$ if $i_1 = \cdots = i_k$, and $\delta_{i_1 \cdots i_k} = 0$ otherwise. There are some results on the research of tensors in [1–3].

For the nonzero vector $\alpha_j \in \mathbb{R}^{n_j}$ $(j = 1, \ldots, k)$, let $(\alpha_j)_i$ be the i-th component of α_j. The *Segre outer product* of $\alpha_1, \ldots, \alpha_k$, denoted by $\alpha_1 \otimes \cdots \otimes \alpha_k$, is called the rank-one tensor $\mathcal{A} = (a_{i_1 \cdots i_k})$ with entries $a_{i_1 \cdots i_k} = (\alpha_1)_{i_1} \cdots (\alpha_k)_{i_k}$ (see [4]). The rank of a tensor $\mathcal{A} \in \mathbb{R}^{n_1 \times \cdots \times n_k}$, denoted by $\mathrm{rank}(\mathcal{A})$, is the smallest r such that $\mathcal{A}$ can be written as a sum of r rank-one tensors as follows:

$$\mathcal{A} = \sum_{j=1}^{r} \alpha_1^j \otimes \cdots \otimes \alpha_k^j, \tag{1.1}$$

where $\alpha_i^j \neq 0$ and $\alpha_i^j \in \mathbb{R}^{n_i}$, $i = 1, \ldots, k$, $j = 1, \ldots, r$ (see [1,4]).

For the vector $x = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$ and a k-order n-dimension tensor $\mathcal{A}$, $\mathcal{A}x^{k-1}$ is an n-dimension vector whose i-th component is

$$(\mathcal{A}x^{k-1})_i = \sum_{i_2, \ldots, i_k \in [n]} a_{i i_2 \cdots i_k} x_{i_2} x_{i_3} \cdots x_{i_k},$$

where $i \in [n]$ (see [2]).

In [5] Shao defines the *general tensor product*. For n-dimension tensors $\mathcal{A} = (a_{i_1 \cdots i_m})$ and $\mathcal{B} = (b_{i_1 \cdots i_k})$ $(m \geq 2,\ k \geq 1)$, their product is an $(m-1)(k-1)+1$-order tensor with entry

$$(\mathcal{A} \cdot \mathcal{B})_{i \alpha_1 \cdots \alpha_{m-1}} = \sum_{i_2, \ldots, i_m \in [n]} a_{i i_2 \cdots i_m} b_{i_2 \alpha_1} \cdots b_{i_m \alpha_{m-1}},$$

where $i \in [n]$, $\alpha_1, \ldots, \alpha_{m-1} \in [n]^{k-1}$. And if $\mathcal{A} \cdot \mathcal{B} = \mathcal{I}$, then $\mathcal{A}$ is called an *m-order left inverse* of $\mathcal{B}$ and $\mathcal{B}$ is called a *k-order right inverse* of $\mathcal{A}$ (see [6]). The determinant of a k-order n-dimension tensor $\mathcal{A}$, denoted by $\det(\mathcal{A})$, is the resultant of the system of homogeneous equation $\mathcal{A}x^{k-1} = 0$, where $x \in \mathbb{R}^n$ (see [3]). In [2] Qi researches the determinant of symmetric tensors. In [5] Shao proves that $\det(\mathcal{A})$ is the unique polynomial on the entries of $\mathcal{A}$ satisfying the following three conditions:

(1) $\det(\mathcal{A}) = 0$ if and only if the system of homogeneous equation $\mathcal{A}x^{k-1} = 0$ has a nonzero solution;
(2) $\det(\mathcal{I}) = 1$;
(3) $\det(\mathcal{A})$ is an irreducible polynomial on the entries of $\mathcal{A}$ when the entries $a_{i_1 \cdots i_k}$ $(i_1, \ldots, i_k \in [n])$ of $\mathcal{A}$ are all viewed as independent different variables. If $\det(\mathcal{A}) \neq 0$, then $\mathcal{A}$ is called a *nonsingular tensor*.