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The numerical range of finite order elliptic automorphism composition operators

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ABSTRACT

In this paper we consider the following conjecture, posed by P.S. Bourdon and J.H. Shapiro in [2]: The numerical range of a finite order elliptic automorphism is not a disk, and we show that this is true for a large class of such composition operators.

1. Introduction

Let φ be a holomorphic self-map of the unit disk $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$. The function φ induces the *composition operator* C_φ, defined on the space of holomorphic functions on $\mathbb{U}$ by $C_\varphi f = f \circ \varphi$. The restriction of C_φ to various Banach spaces of holomorphic functions on $\mathbb{U}$ has been an active subject of research for more than three decades and

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it will continue to be for decades to come (see [14,15,4]). Let H^2 denote the *Hardy space* of analytic functions on the open unit disk with square summable Taylor coefficients. In recent years the study of composition operators on the Hardy space has received considerable attention.

In this paper, we will consider the *numerical range* of elliptic composition operators on H^2. The numerical range of a bounded linear operator A on a Hilbert space $\mathcal{H}$ is the set of complex numbers

$$W(A) := \{\langle Ax, x\rangle : x \in \mathcal{H}, \|x\| = 1\},$$

where $\langle ., .\rangle$ denotes the inner product in $\mathcal{H}$.

In [10] and [11], V. Matache determined the shape $W(C_\varphi)$ when the symbol of the composition operator is a monomial or an inner function fixing 0. Also he gave some properties of the numerical range of composition operators in some cases. In [2] the shapes of the numerical range for composition operators induced on H^2 by some conformal automorphisms of the unit disk specially parabolic and hyperbolic were investigated. In [2], Bourdon and Shapiro have considered the question of when the numerical range of a composition operator is a disk centered at the origin and have shown that this happens whenever the inducing map is a non-elliptic conformal automorphism of the unit disk. They also have shown that the numerical range of elliptic automorphism with order 2 is an ellipse with focus at ± 1. In [1], the second author has completed their results by finding the exact value of the major axis of the ellipses. However, for the elliptic automorphisms with finite order $k > 2$, this is yet an open problem.

In [12], Patton showed that the conjecture of Bourdon and Shapiro does in fact hold for composition operators on H^2 with minimal polynomial equal to $z^3 - 1$: the numerical range of such a composition operator is not a disk.

In this paper, we show that the closure of $W(C_\varphi)$ is not a disk, for a large class of finite order elliptic automorphisms φ.

2. Notations and preliminaries

Let $\mathbb{U}$ denote the open unit disk in the complex plane, and the *Hardy space* H^2 the functions $f(z) = \sum_{n=0}^{\infty} \widehat{f}(n) z^n$ holomorphic in $\mathbb{U}$ such that $\sum_{n=0}^{\infty} |\widehat{f}(n)|^2 < \infty$, with $\widehat{f}(n)$ denoting the n-th Taylor coefficient of f. The inner product inducing the norm of H^2 is given by $\langle f, g\rangle := \sum_{n=0}^{\infty} \widehat{f}(n)\overline{\widehat{g}(n)}$. The inner product of two functions f and g in H^2 may also be computed by integration:

$$\langle f, g\rangle = \frac{1}{2\pi i} \int\limits_{\partial\mathbb{U}} f(z)\overline{g(z)} \frac{dz}{z}$$

where $\partial\mathbb{U}$ is positively oriented and f and g are defined a.e. on $\partial\mathbb{U}$ via radial limits.

For each holomorphic self map φ of $\mathbb{U}$ induces on H^2 a *composition operator* C_φ, defined by the equation $C_\varphi f = f \circ \varphi$ $(f \in H^2)$. A consequence of a famous theorem of