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Discrete norms of a matrix and the converse to the expander mixing lemma

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ABSTRACT

We define the *discrete norm* of a complex $m \times n$ matrix A by

$$\|A\|_\Delta := \max_{0 \neq \xi \in \{0,1\}^n} \frac{\|A\xi\|}{\|\xi\|},$$

and show that

$$\frac{c}{\sqrt{\log h(A) + 1}} \|A\| \le \|A\|_\Delta \le \|A\|,$$

where $c > 0$ is an explicitly indicated absolute constant, $h(A) = \sqrt{\|A\|_1 \|A\|_\infty / \|A\|}$, and $\|A\|_1$, $\|A\|_\infty$, and $\|A\| = \|A\|_2$ are the induced operator norms of A. Similarly, for the *discrete Rayleigh norm*

$$\|A\|_P := \max_{\substack{0 \neq \xi \in \{0,1\}^m \\ 0 \neq \eta \in \{0,1\}^n}} \frac{|\xi^t A \eta|}{\|\xi\| \|\eta\|}$$

we prove the estimate

$$\frac{c}{\log h(A) + 1} \|A\| \le \|A\|_P \le \|A\|.$$

These estimates are shown to be essentially best possible.
As a consequence, we obtain another proof of the (slightly sharpened and generalized version of the) converse to the

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expander mixing lemma by Bollobás–Nikiforov and Bilu–Linial.

1. Summary of results

For a complex matrix A with n columns, we define the *discrete norm* of A by

$$\|A\|_\Delta := \max_{0\neq\xi\in\{0,1\}^n} \frac{\|A\xi\|}{\|\xi\|},$$

where the maximum is over all non-zero n-dimensional binary vectors ξ, and $\|\cdot\|$ denotes the usual Euclidean vector norm. Recalling the standard definition of the induced operator L^2-norm

$$\|A\| := \sup_{0\neq x\in\mathbb{C}^n} \frac{\|Ax\|}{\|x\|},$$

we see immediately that $\|A\|_\Delta \leq \|A\|$, and one can expect that, moreover, the two norms are not far from each other.

1.1. Norm estimates

Our first goal is to establish a result along the lines just indicated; to state it, we introduce the notion of a *height* of a matrix.

For $p \in [1,\infty]$, let $\|A\|_p$ denote the induced operator L^p-norm of the matrix A:

$$\|A\|_p := \sup_{0\neq x\in\mathbb{C}^n} \frac{\|Ax\|_p}{\|x\|_p},$$

where n is the number of columns of A. We are actually interested in the following three special cases: the *column norm* $\|A\|_1$, which can be equivalently defined as the largest absolute column sum of A; the *row norm* $\|A\|_\infty$, which is the largest absolute row sum of A; and the Euclidean norm $\|A\|_2$, commonly denoted simply by $\|A\|$. These three norms are known to be related by the inequality

$$\|A\|^2 \leq \|A\|_1\|A\|_\infty, \tag{1}$$

which can be obtained as a particular case of the Riesz–Thorin theorem, or proved directly, using basic properties of matrix norms (in particular, sub-multiplicativity of the L^1-norm):

$$\|A\|^2 = \|A^*A\| \leq \|A^*A\|_1 \leq \|A^*\|_1\|A\|_1 = \|A\|_\infty\|A\|_1.$$