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Half turns in characteristic 2

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ABSTRACT

Let V be a nonsingular quadratic space over a field K of characteristic 2. We show that if $n > 4$, then every element π in the special orthogonal group $\mathrm{SO}(V)$ is a product of an even number of half turns. If $\dim B(\pi) = 2k$, then the length of π with respect to the half turns is k, $k+1$ or $k+2$.

1. Introduction

Let V be an n-dimensional vector space over a field K and let $q : V \longrightarrow K$ be a quadratic form. An element of the orthogonal group $\mathrm{O}(V)$ is called *half turn* if it is a product of two distinct commuting reflections. If the characteristic of K is distinct

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from 2 and $n \geq 3$, Artin discovered that every element in the special orthogonal group $\mathrm{SO}(V)$ is a product of n or fewer half turns (see [1, Thm. 3.22]). The minimal number of half turns needed to write an element of the special orthogonal group as product of half turns was determined by Ishibashi (see [5]) for regular spaces and in general by Ellers (see [2]). It remains to study the case of characteristic 2. Half turns are related to Siegel transformations, which were studied for some fields of characteristic 2 in [3]. Here we consider fields of characteristic 2 with only one restriction: we exclude the case in which $\dim V = 4$, $\mathrm{ind} V = 2$, and K has only two elements; in this case the orthogonal group is not generated by reflections. We show that also in characteristic 2, if $n \geq 4$, every element of $\mathrm{SO}(V)$ ($\mathrm{SO}(V) \neq \mathrm{O}^+(4,2)$) is a product of an even number of half turns (see Theorem 3.3). We also show that if $\pi \in \mathrm{SO}(V)$ such that $\dim B(\pi) = 2k$ and $n \geq 6$, then π is a product of k or $k+1$ or $k+2$ half turns (see Theorem 3.12).

2. Notation

Let V be a vector space of dimension n over a field K of characteristic 2. Let $q : V \longrightarrow K$ be a quadratic form and $b_q : V \times V \longrightarrow K$ the symmetric bilinear form defined by $b_q(x, y) := q(x + y) - q(x) - q(y)$ for $x, y \in V$. We assume that the bilinear form b_q is *nondegenerate* (alias *nonsingular*), i.e. $b_q(x, y) = 0$ for all $y \in V$ implies $x = 0$. The dimension of V is even because the bilinear form b_q is *alternating*, i.e. $b_q(x, x) = 0$ for all $x \in V$. Two vectors $v, w \in V$ are called *perpendicular*, $v \perp w$, if and only if $b_q(v, w) = 0$. A vector $v \in V$ is called *isotropic* if $b_q(v, v) = 0$ and *singular* if $q(v) = 0$. Of course in characteristic 2 every vector is isotropic. Let W be a subspace of V. Then W is called *totally isotropic* if $b_q(u, w) = 0$ for all $u, w \in W$ and *totally singular* if $q(w) = 0$ for all $w \in W$. A totally singular subspace is also totally isotropic, but the converse is not necessarily true. Since b_q is nondegenerate, the dimension of any totally singular (resp. totally isotropic) subspace of V is less than or equal to $n/2$. The common dimension of the maximal totally singular subspaces of V is called the *index* of V and is denoted by $\mathrm{ind} V$. The *orthogonal complement* of W is defined by $W^\perp := \{x \in V \mid b_q(x, w) = 0$ for all $w \in W\}$. The subspaces $\mathrm{rad}\, W = W \cap W^\perp$ and $SW = \{x \in \mathrm{rad}\, W \mid \; q(x) = 0\}$ are called the *radical* of W and the *singular* of W, respectively. The space W is said to be *nonsingular* if $\mathrm{rad}\, W = 0$. Let $\mathrm{GL}(V)$ be the general linear group of V. The *orthogonal group* with respect to q is defined by

$$\mathrm{O}(V) := \{\pi \in \mathrm{GL}(V) \mid q(\pi(x)) = q(x) \quad \text{for all} \quad x \in V\}.$$

The *residual space* or *path* $B(\pi)$ of an element π in $\mathrm{O}(V)$ is the space

$$B(\pi) := \{\pi(x) - x \mid x \in V\}.$$

For every element $\pi \in \mathrm{O}(V)$, we define the *fix* $F(\pi) = \{x \in V \mid \pi(x) = x\}$. Clearly, $B(\pi)^\perp = F(\pi)$ and $n = \dim B(\pi) + \dim F(\pi)$ for every transformation $\pi \in \mathrm{O}(V)$.