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# On mappings preserving the sharp and star orders

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ABSTRACT

The present paper is devoted to the study of linear maps preserving certain relations, such as the sharp partial order and the star partial order in semisimple Banach algebras and $C^*$-algebras.



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## 1. Introduction and background

Let $A$ be a Banach algebra. Recall that an element $a \in A$ is *regular* if there is $b \in A$ such that $aba = a$. For a regular element $a \in A$, the set

$$a\{1\} = \{x \in A: axa = a\}$$

consists of all $\{1\}$-*inverses* or *inner inverses* of $a$. Notice that if $x$ is a $\{1\}$-inverse of $a$, then $ax$ and $xa$ are idempotents. A $\{1,2\}$-*inverse* or *generalized inverse* of $a$, is a $\{1\}$-inverse of $a$ that is a solution of the equation $xax = x$, that is, it is an element $b \in A$ such that $aba = a$ and $bab = b$.

Note that the condition $x \in a\{1\}$ ensures the existence of a generalized inverse of $a$: in such case, $b = xax$ fulfills the previous identities.

For an element $a$ in $A$, let us consider the left and right multiplication operators $L_a : x \mapsto ax$ and $R_a : x \mapsto xa$, respectively. If $a$ is regular, then so are $L_a$ and $R_a$, and thus their ranges $aA = L_a(A)$ and $Aa = R_a(A)$ are both closed. The unique generalized inverse of $a$ that commutes with $a$ is called the *group inverse* of $a$, whenever it exists. In this case $a$ is said to be *group invertible* and its group inverse is denoted by $a^\sharp$. The set of all group invertible elements of $A$ is denoted by $A^\sharp$.

Even though regularity can be defined in general Banach algebras, this notion has been mostly studied in C*-algebras. Harte and Mbekhta proved in [20] that an element $a$ in a unital C*-algebra $A$ is regular if and only if $aA$ is closed. Given $a$ and $b$ in a C*-algebra $A$, we shall say that $b$ is a *Moore–Penrose inverse* of $a$ if $b$ is a generalized inverse of $a$ and $ab$ and $ba$ are selfadjoint. It is known that every regular element $a$ in $A$ has a unique Moore–Penrose inverse that will be denoted by $a^\dagger$ [20]. We write $A^\dagger$ for the set of regular elements in the C*-algebra $A$.

Let $M_n(\mathbb{C})$ be the algebra of all $n \times n$ complex matrices. On $M_n(\mathbb{C})$ there are many partial orders, which have been well studied (see [17,21,22,28–30]). The *star partial order* on $M_n(\mathbb{C})$ was introduced by Drazin in [17], as follows:

$$A \leq_* B \qquad \text{if and only if} \qquad A^*A = A^*B \text{ and } \ AA^* = BA^*,$$

where as usual $A^*$ denotes the conjugate transpose of $A$. It was proved that $A \leq_* B$ if and only if $A^\dagger A = A^\dagger B$ and $AA^\dagger = BA^\dagger$. Baksalary and Mitra introduced in [4] the *left-star* and *right-star* partial order on $M_n(\mathbb{C})$, as

$$A* \leq B \qquad \text{if and only if} \qquad A^*A = A^*B \text{ and } \ \operatorname{Im} A \subseteq \operatorname{Im} B,$$

and

$$A \leq *B \qquad \text{if and only if} \qquad AA^* = BA^* \text{ and } \ \operatorname{Im} A^* \subseteq \operatorname{Im} B^*,$$

respectively. Moreover, $A \leq_* B$ if and only if $A* \leq B$ and $A \leq *B$.