Linear Algebra and its Applications 483 (2015) 293–308

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Preserving problems of geodesic-affine maps and related topics on positive definite matrices ☆

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A R T I C L E I N F O

Article history:
Received 17 December 2014
Accepted 10 June 2015
Available online 23 June 2015
Submitted by R. Bhatia

MSC:
47B49
15B48
53C22

Keywords:
Positive definite matrices
Riemannian metric
Geodesic-affine map
Relative entropy
Preserving problems

A B S T R A C T

Based on affine maps in geometry, we study the geodesic-affine maps on Riemannian manifolds $\mathbb{P}_n$ of complex positive definite matrices that are induced by different so-called kernel functions. In this article, we are going to describe the structure of all continuous bijective geodesic-affine maps on these manifolds. We also prove that geodesic distance isometries are geodesic-affine maps. Moreover, the forms of all bijective maps which preserve norms of geodesic correspondence are characterized. Indeed, these maps are special examples of geodesic-affine maps.

☆ The first author was supported by the "Lendület" Program (LP2012-46/2012) of the Hungarian Academy of Sciences and by the Hungarian Scientific Research Fund (OTKA) Grant NK81402. The second author was supported by the Taiwan NSC Grant 102-2811-M-110-018. The third author was supported NSFC grant (No. 11171126).

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http://dx.doi.org/10.1016/j.laa.2015.06.009
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1. Introduction

Let $\mathbb{P}_n$ be the set of all $n \times n$ complex positive definite matrices. An affine map ϕ on $\mathbb{P}_n$ is a map satisfying that

$$\phi((1-t)A + tB) = (1-t)\phi(A) + t\phi(B), \qquad t \in [0,1],\ A, B \in \mathbb{P}_n.$$

Namely, ϕ maps the segment joining A, B onto segment joining $\phi(A), \phi(B)$. It is known that the set $\mathbb{P}_n$ can be equipped with certain Riemannian structures via different Riemannian metrics and the geodesics in these manifolds play the same role as segments. Therefore we can define the geodesic-affine map ϕ as a map sending the points of geodesic $\gamma_{A,B}(t)$ joining A, B into geodesic $\gamma_{\phi(A),\phi(B)}(t)$ joining $\phi(A), \phi(B)$, i.e.,

$$\phi(\gamma_{A,B}(t)) = \gamma_{\phi(A),\phi(B)}(t), \qquad t \in [0,1],\ A, B \in \mathbb{P}_n.$$

In addition, we say that ϕ preserves norms of geodesic correspondence if

$$\|\gamma_{A,B}(t)\| = \|\gamma_{\phi(A),\phi(B)}(t)\|, \qquad t \in [0,1],\ A, B \in \mathbb{P}_n.$$

In this paper, we will study the geodesic-affine maps in $\mathbb{P}_n$. The paper is organized as follows. In the present section we introduce the necessary notion and notation which will be used throughout the paper. In Section 2, we characterize the structural results of the geodesic-affine maps on these Riemannian manifolds. Section 3 is concerned with the relationship between geodesic-affine maps and geodesic distance isometries. We will show that every geodesic distance isometry is a geodesic-affine map, and the converse is not always true. In Section 4, the forms of maps which preserve the norm of all points of geodesics in Riemannian manifolds corresponding to different kinds of kernel functions will be studied. We will show that these maps are also geodesic distance isometries, and hence are geodesic-affine maps.

Throughout this paper we denote by $\mathbb{M}_n$ the set of all $n \times n$ complex matrices and by $\mathbb{H}_n$ the set of all $n \times n$ Hermitian matrices. The elements of $\mathbb{P}_n$ form an open subset of $\mathbb{H}_n$ and it can be equipped with a Riemannian structure such that the tangent space at any foot point $D \in \mathbb{P}_n$ is identified with $\mathbb{H}_n$. A Riemannian metric $K_D\colon \mathbb{H}_n \times \mathbb{H}_n \to [0,\infty)$ is a family of inner products on $\mathbb{H}_n$ depending smoothly on the foot point D. If $\varphi\colon (0,\infty) \times (0,\infty) \to (0,\infty)$ is a so-called kernel function, i.e., it is a symmetric function ($\varphi(x,y) = \varphi(y,x)$, for every $x, y \in (0,\infty)$), and is smooth in its both variables, and D has the spectral decomposition $\sum_{i=1}^{n} \lambda_i P_i$, then a Riemannian metric can be defined by

$$K_D^{\varphi}(H,K) := \sum_{i,j=1}^{n} \varphi(\lambda_i,\lambda_j)^{-1} \operatorname{tr} P_i H P_j K, \quad D \in \mathbb{P}_n,\ H, K \in \mathbb{H}_n, \tag{1.1}$$