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Some new considerations about double nested graphs

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ABSTRACT

In the set of all connected graphs with fixed order and size, the graphs with maximal index are nested split graphs, also called threshold graphs. It was recently (and independently) observed in Bell et al. (2008) [3] and Bhattacharya et al. (2008) [4] that double nested graphs, also called bipartite chain graphs, play the same role within class of bipartite graphs. In this paper we study some structural and spectral features of double nested graphs. In studying the spectrum of double nested graphs we rather consider some weighted nonnegative matrices (of significantly less order) which preserve all positive eigenvalues of former ones. Moreover, their inverse matrices appear to be tridiagonal. Using this fact we provide several new bounds on the index (largest eigenvalue) of double nested graphs, and also deduce some bounds on eigenvector components for the index. We conclude

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the paper by examining the questions related to main versus non-main eigenvalues.

1. Introduction

Let $G = (V(G), E(G))$ be an undirected simple graph, i.e. a finite graph without loops or multiple edges. $V(G)$ is its vertex set, while $E(G)$ its edge set. The order of G is denoted by ν $(= |V(G)|)$, and its size by ϵ $(= |E(G)|)$. We write $u \sim v$ whenever vertices $u, v \in V(G)$ are adjacent, and denote by uv the corresponding edge.

Given a graph G, $A(G)$ denotes the $(0,1)$-adjacency matrix of G. The polynomial $P(x; G) = \det(xI - A(G))$ is called the *characteristic polynomial* of G. Its roots comprise the *spectrum* of G, denoted by $\mathrm{Sp}(G)$. Since $A(G)$ is symmetric, its spectrum is real, and in general it is a multiset containing ν non-necessarily distinct *eigenvalues*. So let

$$\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_\nu(G)$$

be the corresponding eigenvalues (given in non-increasing order). Recall, as well known, $\lambda_1(G) > \lambda_2(G)$ whenever G is connected. Further on, if not told otherwise, we will consider only connected graphs. The largest eigenvalue of G, denoted by $\rho(G)$, is called the *spectral radius* of G (or, for short, its *index*). For a given $\lambda \in \mathrm{Sp}(G)$, $m(\lambda; G)$ denotes the *multiplicity* of λ in G. Since $A(G)$ is symmetric, the algebraic and geometric multiplicities of λ coincide. If $m(\lambda; G) = 1$, then λ is a *simple eigenvalue* of G.

The equation $A\mathbf{x} = \lambda\mathbf{x}$ is called the *eigenvalue equation* of A, or of a labelled graph G, if $A = A(G)$. For a fixed $\lambda \in \mathrm{Sp}(G)$, a non-trivial solution $\mathbf{x} = (x_1, x_2, \ldots, x_\nu)^T$ of the eigenvalue equation is a λ-eigenvector of a labelled graph G. In particular, if $\lambda = \rho(G)$, then the corresponding vector, with positive coordinates, is called a *principal eigenvector* of G. In the scalar form, for any $\lambda \in \mathrm{Sp}(G)$, the eigenvalue equation reads:

$$\lambda x_u = \sum_{v \sim u} x_v,$$

where $u \in V(G)$. The null space of $A(G) - \lambda I$ is called the eigenspace of G and is denoted by $\mathcal{E}(\lambda; G)$. Note also that any $\mathbf{x} \in \mathcal{E}(\lambda; G)$ can be interpreted as a mapping $\mathbf{x} : V(G) \to \mathbb{R}$. So, for any $v \in V(G)$, $\mathbf{x}(v)$ and x_v can be identified, and considered as vertex weights (with respect to $\mathbf{x}$).

Finally, recall that an eigenvalue μ of G is *main* if $\mathcal{E}(\mu; G)$ is not orthogonal to $\mathbf{j}$, i.e. the all-1 vector; otherwise it is *non-main*.

For all other notions (and notation) from graph theory, including spectral graph theory, the reader is referred to the book [8]. The same notation will be adopted for matrices, by passing from the adjacency matrix to an arbitrary (symmetric) matrix.