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Tucker's theorem for almost skew-symmetric matrices and a proof of Farkas' lemma

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A real square matrix A is said to be almost skew-symmetric if its symmetric part has rank one. In this article certain fundamental questions on almost skew-symmetric matrices are considered. Among other things, necessary and sufficient conditions on the entries of a matrix in order for it to be almost skew-symmetric are presented. Sums and subdirect sums are studied. Certain new results for the Moore–Penrose inverse of an almost skew-symmetric matrix are proved. An interesting analogue of Tucker's theorem for skew-symmetric matrices is derived for almost skew-symmetric matrices. Surprisingly, this analogue leads to a proof of Farkas' lemma.

1. Introduction

In this article, we study matrices $A \in \mathbb{R}^{n \times n}$ which have the property that their symmetric part is of rank one. Such matrices are called almost skew-symmetric. The motivation for this notion seems to have come from tournament matrices and their

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extensions. The spectra of almost skew-symmetric matrices were also considered in the literature. We refer the reader to [8] and the references cited therein, for these details. There, certain interesting results on eigenvalues and numerical range of almost skew-symmetric matrices are also derived. In [9], the authors study inheritance properties of almost skew-symmetry by the Schur complement and a generalized principal pivot transform.

The outline of the present work is as follows: This introductory section is followed by a short section on the preliminary results that will be required in the rest of the article. This includes some terminology. In Section 3, first we derive a fundamental result on the structure of an almost skew-symmetric matrix by giving certain conditions for the entries of the matrix. This is presented in Theorem 3.3. We then present a characterization for the sum of two almost skew-symmetric matrices to be almost skew-symmetric, in Theorem 3.4. A similar question for the subdirect sum is considered next. An answer is provided in Theorem 3.5. A partial converse is proved in Theorem 3.6. Section 4 presents generalizations of some results for invertible almost skew-symmetric matrices to the case of the Moore–Penrose inverse. These are given in Theorem 4.1, Theorem 4.2 and Theorem 4.3. For a skew-symmetric matrix A, a result of Tucker [11, Theorem 5] asserts that, there exists a nonnegative vector u such that Au is nonnegative and the vector $Au + u$ is strictly positive. This is widely referred to as Tucker's theorem in the literature. Broyden has shown that this theorem is equivalent to Farkas' lemma, a very well known theorem of alternative. In Section 5, a version of Tucker's theorem for almost skew-symmetric matrices is presented in Theorem 5.2. Quite surprisingly, we are able to prove Farkas' lemma from this result. We conclude the article by mentioning certain preservers of almost skew-symmetry.

2. Preliminaries

Let $\mathbb{R}^{n\times n}$ denote the set of all $m \times n$ matrices over the real numbers. For $A \in \mathbb{R}^{n\times n}$ let $S(A)$, $K(A)$, $R(A)$, $N(A)$ and $rk(A)$ denote the symmetric part of A ($\frac{1}{2}(A+A^t)$), the skew-symmetric part of A ($\frac{1}{2}(A-A^t)$), the range space of A, the null space of A and the rank of A. For $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ let $diag(A)$ denote the column vector of diagonal entries of A: $diag(A) = (a_{11}, a_{22}, \ldots, a_{nn})^t$. The Moore–Penrose inverse of a matrix $A \in \mathbb{R}^{m\times n}$ is the unique matrix $X \in \mathbb{R}^{n\times m}$ satisfying $A = AXA$, $X = XAX$, $(AX)^T = AX$ and $(XA)^T = XA$ and is denoted by $A^\dagger$. The group inverse of a matrix $A \in \mathbb{R}^{n\times n}$, if it exists, is the unique matrix $X \in \mathbb{R}^{n\times n}$ satisfying $A = AXA$, $X = XAX$ and $AX = XA$ and is denoted by $A^\#$. If A is nonsingular, then $A^{-1} = A^\dagger = A^\#$. Recall that $A \in \mathbb{R}^{n\times n}$ is called range-symmetric if $R(A) = R(A^t)$. If A is range-symmetric, then $A^\dagger = A^\#$ [2, Theorem 4, p. 157]. $A \in \mathbb{R}^{n\times n}$, $n \geq 2$ is called an almost skew-symmetric matrix if $rk(S(A)) = 1$, where $S(A)$ is the symmetric part of A. It follows at once that A is almost skew-symmetric if and only if A^T is almost skew-symmetric. The nonzero eigenvalue of $S(A)$ is denoted by $\delta(A)$. In the remainder of the article we assume that $\delta(A) > 0$; otherwise, our results are applicable to $-A$. It follows that if A is an almost skew-symmetric matrix then