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Can Sylvester's determinantal identity, equivalently Muir's law of extensible minors be generalized?

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We show that Sylvester's classical determinantal identity is equivalent to its generalizations given by the Mühlbach–Gasca–(López-Carmona)–Ramírez identity and the Beckermann–Mühlbach identity. We also show that the generating extension principles associated with each of these three identities (Muir's law of extensible minors, Mühlbach's extension principle and the extension principle given in Camargo, 2013 [3]) are equivalent and we use this fact to conclude that all these six statements (and also Jacobi's identity on the minors of the adjugate and Cayley's law of complementaries) are actually equivalent. These findings lead naturally to enquire whether there would exist a generalization of Sylvester's identity (or an extension principle) which could not be derived from Sylvester's identity itself.

1. Introduction

In this Note we explore the equivalences between the six statements depicted in Fig. 1 (their precise description is given in Section 2). In the left column, we have Sylvester's

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Sylvester's identity (**S**)	⟵	Muir's principle (**ML**)
∩		∩
Mühlbach–Gasca–(Lópes-Carmona)–Ramírez's identity (**MG**)	⟵	Mühlbach's principle (**MEP**)
∩		∩
Mühlbach–Beckermann's identity (**BM**)	⟵	(**CEP**)

Fig. 1. Identities of Sylvester's type and corresponding extension principles.

classical determinantal identity (**S**) [2,6,13], its generalizations found by Mühlbach, Gasca, López-Carmona and Ramírez (**MG**) [8,10] and a second generalization presented by Beckerman and Mühlbach (**BM**) [1]. Each of these three identities can be viewed as an extension of Leibniz's definition of the determinant by means of the corresponding extension principle in the right column of Fig. 1.

In our previous paper [3] we presented the extension principle (**CEP**) which generalizes both Muir's law of extensible minors (**ML**) [12] and Mühlbach's extension principle (**MEP**) [10], and we established the connection between (**CEP**) and (**BM**) which was missing to complete the diagram in Fig. 1 (the meaning of the arrows in this diagram is explained in detail in [3]). Surprisingly, we found later that (**CEP**) and (**MEP**) are equivalent [4] and, as a corollary, we established the equivalence between (**MG**) and (**BM**).

In this Note we show that Sylvester's classical determinantal identity (**S**) is equivalent to the Mühlbach–Gasca–(López-Carmona)–Ramírez identity (**MG**). As a direct consequence, we have that (**S**), (**MG**) and (**BM**) are equivalent. We also show that (**ML**) and (**MEP**) are equivalent and we use this information (and also the further results in [4]) to show that all the six statements in Fig. 1 (and also Jacobi's identity on the minors of the adjugate and Cayley's law of complementaries, which are stated in Section 5) are actually equivalent.

These facts lead naturally to ask whether there would exist a generalization of Sylvester's identity (or an extension principle) which could not be derived from Sylvester's identity itself. As pointed out by Karapiperi, Russo and Redivo-Zaglia [17], other generalizations of Sylvester's identity exist [11,16], but they are clearly grounded on Sylvester's identity as well.

1.1. Notation

Throughout this paper, n, m and p will denote fixed positive integers with $m < \min\{n, p\}$. An ordered sequence $(i_1, i_2, \ldots, i_r)$ of positive integers, $i_1 \leq i_2 \leq \ldots \leq i_r \leq p$, will be called an index list. Operations are defined between index lists as for sets, but the results are understood to be ordered. E.g., if $I = (i_1, i_2, \ldots, i_{r_i})$ and $J = (j_1, j_2, \ldots, j_{s_j})$ are two index lists such that $i_{r_i} < j_1$, then $I \overset{\bullet}{\cup} J$ will denote the concatenated index list $(i_1, i_2, \ldots, i_{r_i}, j_1, j_2, \ldots, j_{s_j})$ and $\#I$ will denote the cardinality of I. For an index list