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Uniqueness problem on numerical ranges of 3-by-3 companion matrices

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ABSTRACT

For any two 3-by-3 companion matrices A and B with identical numerical ranges, we give a necessary and sufficient condition for $A = B$ in terms of the shapes of numerical ranges and the locations of eigenvalues. In addition, all distinct 3-by-3 companion matrices with identical numerical ranges can be obtained precisely.

1. Introduction

Let A be an n-by-n complex matrix. Then the *numerical range* of A, $W(A)$, is defined as $\{\langle Ax, x\rangle\colon\ x \in \mathbb{C}^n,\ \|x\| = 1\}$, where $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ denote the standard inner product and its associated norm in $\mathbb{C}^n$, respectively. It is well known that $W(A)$ is a nonempty compact convex subset of the complex plane. Other properties of the numerical range can be found in [4, Chapter 1].

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An n-by-n *companion matrix* A is one of the form

$$\begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ -a_n & -a_{n-1} & \cdots & -a_2 & -a_1 \end{bmatrix}. \tag{1.1}$$

It is known that the characteristic and minimal polynomials of such an A are both equal to $z^n + a_1 z^{n-1} + \cdots + a_{n-1}z + a_n$.

Our purpose of this paper is to solve the *uniqueness problem* on numerical ranges of 3-by-3 companion matrices: "assume that A and B are 3-by-3 companion matrices. Can we infer from $W(A) = W(B)$ that $A = B$?" In general, this is not true (cf. [3, Example 2.1]). However, if A and B are restricted to be reducible or $W(A) = W(B)$ is not a non-circular elliptic disc, then $A = B$ (cf. [2, Theorem 2.10] and [3, Theorem 2.2]). These results enable us to concentrate on when a 3-by-3 irreducible companion matrix can be determined completely by its elliptic numerical range.

For any two complex numbers z_1 and z_2, we make a minor modification of the existence theorem (cf. [1, Theorem 5.1]) that there exists a 3-by-3 irreducible companion matrix whose numerical range is an elliptic disc with foci z_1 and z_2 (Theorem 2.6). In particular, if $z_1 + z_2$ and $z_1 z_2$ are real numbers, then all 3-by-3 irreducible companion matrices can be found explicitly (Proposition 2.9) which generalizes [1, Theorem 3.1].

We also give a criterion in terms of eigenvalues for 3-by-3 irreducible companion matrices with identical elliptic numerical ranges (Theorem 2.10). At the end of the paper, we summarize essential results to solve the uniqueness problem thoroughly (Theorem 2.17): for any two 3-by-3 companion matrices A and B with identical numerical ranges, $A = B$ if and only if either their numerical range is not an elliptic disc or an elliptic disc with foci z_1 and z_2, where z_1 and z_2 can be described geometrically and their absolute values satisfy a certain inequality. Furthermore, all distinct 3-by-3 companion matrices with identical numerical ranges can be obtained precisely.

2. Companion matrices

We start by reviewing the following criterion for a 3-by-3 matrix whose numerical range is an elliptic disc (cf. [5, Theorem 2.4]).

Theorem 2.1. *Let A be a 3-by-3 matrix with eigenvalues z_1, z_2 and z_3. Then the numerical range $W(A)$ is an elliptic disc with foci z_1 and z_2 if and only if*

(a) $d = (\mathrm{tr}(A^*A) - \sum_{j=1}^{3} |z_j|^2)^{1/2} > 0$,
(b) $z_3 = \mathrm{tr}\, A + (1/d^2)(\sum_{j=1}^{3} |z_j|^2 z_j - \mathrm{tr}(A^*A^2))$, *and*
(c) *z_3 lies inside the elliptic disc with foci z_1, z_2 and minor axis of length d.*