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Matrices commuting with a given normal tropical matrix

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ABSTRACT

Consider the space M_n^{nor} of square normal matrices $X = (x_{ij})$ over $\mathbb{R} \cup \{-\infty\}$, i.e., $-\infty \leq x_{ij} \leq 0$ and $x_{ii} = 0$. Endow M_n^{nor} with the tropical sum $\oplus$ and multiplication $\odot$. Fix a real matrix $A \in M_n^{nor}$ and consider the set $\Omega(A)$ of matrices in M_n^{nor} which commute with A. We prove that $\Omega(A)$ is a finite union of alcoved polytopes; in particular, $\Omega(A)$ is a finite union of convex sets. The set $\Omega^A(A)$ of X such that $A \odot X = X \odot A = A$ is also a finite union of alcoved polytopes. The same is true for the set $\Omega'(A)$ of X such that $A \odot X = X \odot A = X$.
A topology is given to M_n^{nor}. Then, the set $\Omega^A(A)$ is a neighborhood of the identity matrix I. If A is strictly normal, then $\Omega'(A)$ is a neighborhood of the zero matrix. In one case, $\Omega(A)$ is a neighborhood of A. We give an upper bound for the dimension of $\Omega'(A)$. We explore the relationship between the polyhedral complexes span A, span X and span(AX), when A and X commute. Two matrices, denoted $\underline{A}$ and $\overline{A}$, arise from A, in connection with $\Omega(A)$. The geometric meaning of them is given in detail, for one example. We produce examples of matrices which commute, in any dimension.

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1. Introduction

Let $n \in \mathbb{N}$ and K be a field. Fix a matrix $A \in M_n(K)$ and consider $K[A]$, the algebra of polynomial expressions in A. In classical mathematics, the set $\Omega(A)$ of matrices commuting with A is well-known: $\Omega(A)$ equals $K[A]$ if and only if the characteristic and minimal polynomials of A coincide. Otherwise, $K[A]$ is a proper linear subspace of $\Omega(A)$; see [27, Chap. VII].

In this paper we study the analog of $\Omega(A)$ in the tropical setting. Moreover, we restrict ourselves to square *normal* matrices over $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty\}$, i.e., matrices $A = (a_{ij})$ with $a_{ii} = 0$ and $-\infty \le a_{ij} \le 0$, for all i, j. The set of all such matrices, endowed with the tropical operations $\oplus = \max$ and $\odot = +$, is denoted M_n^{nor}.

For any $r \in \mathbb{R}_{\le 0}$, the half-line $[-\infty, r) := \{x : -\infty \le x < r\}$ is open in $\overline{\mathbb{R}}_{\le 0}$ with the usual interval topology. A Cartesian product of such half-lines is open in $\overline{\mathbb{R}}_{\le 0}^{n^2-n}$ with the usual product topology. The half-line $(r, 0] := \{x : r < x \le 0\}$ is open in $\overline{\mathbb{R}}_{\le 0}$. A Cartesian product of such half-lines is open in $\overline{\mathbb{R}}_{\le 0}^{n^2-n}$.

The set M_n^{nor} can be identified with $\overline{\mathbb{R}}_{\le 0}^{n^2-n}$ and, via this identification, M_n^{nor} gets a *topology*. Consider a matrix $X \in M_n^{nor}$ and a subset $V \subseteq M_n^{nor}$. We say that V is a *neighborhood* of X if there exists an open subset $U \subseteq M_n^{nor}$ such that $X \in U \subseteq V$ (we do not require V to be open).

Let $\Omega(A)$ be the subset of matrices commuting with a given real matrix A, i.e., $X \in M_n^{nor}$ such that $A \odot X = X \odot A$. The tropical analog of $K[A]$ inside M_n^{nor} is the set $\mathcal{P}(A)$ of tropical powers of A. In general, $\Omega(A)$ is larger than $\mathcal{P}(A)$ (see Proposition 1).

Our new results are gathered in Sections 3, 4 and 5. In Section 3 we prove that

$$\Omega(A) = \bigcup_w \Omega_w(A)$$

is a finite union of alcoved polytopes (see Corollary 5). In particular, $\Omega(A)$ is a finite union of convex sets.

Two important subsets of $\Omega(A)$ are

$$\Omega^A(A) = \{X \in \Omega(A) : X \odot A = A \odot X = A\}$$

and

$$\Omega'(A) = \{X \in \Omega(A) : X \odot A = A \odot X = X\}.$$

Both are finite unions of alcoved polytopes (see Theorems 9 and 12). Moreover, $\Omega^A(A)$ is a neighborhood (not necessarily open) of the identity matrix I. If A is strictly normal, then $\Omega'(A)$ is a neighborhood of the zero matrix 0 (see Propositions 7 and 8).

The study of $\Omega^A(A)$ and $\Omega'(A)$ lead us to two matrices arising from A, denoted $\underline{A}$ and $\overline{A}$, and we prove