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Spectral variation bounds in hyperbolic geometry

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We derive new estimates for distances between optimal matchings of eigenvalues of non-normal matrices in terms of the norm of their difference. We introduce and estimate a hyperbolic metric analogue of the classical spectral-variation distance. The result yields a qualitatively new and simple characterization of the localization of eigenvalues. Our bound improves on the best classical spectral-variation bounds due to Krause if the distance of matrices is sufficiently small and is sharp for asymptotically large matrices. Our approach is based on the theory of model operators, which provides us with strong resolvent estimates. The latter naturally lead to a Chebyshev-type interpolation problem with finite Blaschke products, which can be solved explicitly and gives stronger bounds than the classical Chebyshev interpolation with polynomials. As compared to the classical approach our method does not rely on Hadamard's inequality and immediately generalizes to algebraic operators on Hilbert space.

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1. Introduction

For arbitrary complex $n \times n$-matrices $A, B \in \mathcal{M}_n$ we study distances of optimal matchings of their spectra $\sigma(A), \sigma(B)$. The (Euclidean) *optimal matching distance* [2] of two sets $\{a_i\}_{i=1}^n, \{b_i\}_{i=1}^n \subset \mathbb{C}$ is defined as

$$d_E\left(\{a_i\},\{b_i\}\right) = \min_{\sigma\in S_n} \max_{1\leq i\leq n} \left|a_i - b_{\sigma(i)}\right|,$$

where S_n denotes the group of permutations of n objects. A prototypical spectral variation bound in terms of this distance is of the form

$$d_E\left(\sigma\left(A\right),\sigma\left(B\right)\right) \leq C_n\left(\|A\|+\|B\|\right)^{1-\frac{1}{n}}\|A-B\|^{\frac{1}{n}}, \tag{1}$$

where C_n can only depend on n and $\|\cdot\|$ denotes the usual operator norm. Such estimates have been studied in many articles and books over the last decades, see for example [18,9,7,19,11,3,14,22] and the references therein. Despite considerable effort the best C_n in (1) is still not known, the currently best value seems to be $C_n = \frac{16}{3\sqrt{3}}$ [14].

In this work we present a new approach to spectral variation estimates and derive new bounds that characterize the localization of spectra of non-normal matrices. We introduce a (pseudo-) hyperbolic analogue of the optimal matching distance and derive estimates on this quantity in terms of $\|A\|$, $\|B\|$, $\|A-B\|$ and n. These hyperbolic estimates are generally incomparable to Eq. (1) meaning that there are cases, where they perform better than the previously known bounds, but also cases where they do not, see Fig. 1. As it turns out we can use the hyperbolic estimates to improve the best value for C_n, if $\|A-B\|$ is *small enough* (Corollary 7 below). In the limit of large n our C_n approaches 2, which is optimal.

Our argument is guided by a classical approach due to Phillips [19] that in turn builds on techniques developed by Friedland [9] and Elsner [6]. Phillips reduces the problem of obtaining a good estimate of the form (1) to one of *minimizing* the norm of a resolvent along certain paths in the complex plane. The latter can be accomplished using a classical interpolation theorem due to Chebyshev. Phillips' approach was developed further by Bhatia, Elsner and Krause [3,14] who employed a Hadamard-type inequality [7] (Eq. (2) below) due to Elsner to avoid resolvent estimates. Their approach provided a better estimate for C_n.

On the technical side, our article contains two key innovations to the methods developed in the cited publications. First, we employ recent spectral resolvent estimates [23,15] that are stronger than the Hadamard-type inequality (2) used by Bhatia, Elsner and Krause. These resolvent estimates are derived using an interpolation-theoretic approach to eigenvalue bounds introduced in [15]. Second, our resolvent estimates naturally lead us to a Chebyshev-type interpolation problem with finite Blaschke products that yields quantitatively better estimates as compared to the analogous classical Chebyshev interpolation problem for polynomials. The solution to this interpolation problem was